AN EFFICIENT KRYLOV MODEL REDUCTION APPROACH
FOR THE DIRECT EVALUATION OF
ANALYTICAL FREQUENCY AVERAGE OF TRANSFER FUNCTIONS
IN THE LOW-, MID-, AND HIGH-FREQUENCY RANGES

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ABSTRACT

Analytical expressions for the Gaussian frequency average and frequency variance of transfer functions, of linear dynamic systems are available from the author's previous work [1]. Since these expressions are valid and can be evaluated at any frequency, independently of the system complexity and modal density, and since they also give the frequency average of the energy, they provide a natural framework in which to study the transition from low- to high-frequency ranges. In principle, the analytical expressions would require the knowledge of the modal characteristics of the dynamic system of interest. The fact that modal information is not needed to evaluate the frequency average of transfer functions is highlighted. Rather, the average can be evaluated up to desired precision by using an analytical Krylov model reduction approach. The efficiency of this approach is demonstrated on systems of different dimensions and modal densities. It is further demonstrated that system characteristics that are important at different frequency ranges can be integrated into a single reduced model and that low- to high-frequency ranges analysis can be run with a single model.

1. INTRODUCTION

The responses of dynamic systems are usually treated by two different approaches in low and high frequency ranges: at low frequency, the responses are mostly considered deterministically with well defined modes; at high-frequency, a statistical point of view is used and the system power and energy averages are the values of interest. The latter is based on the fact that under some conditions the statistics of these averages simplify which permits the use of statistical energy analysis (SEA). The mid-frequency range where the systems can no more be treated deterministically for practical or conceptual reasons, while the conditions
of use of SEA are not satisfied is more challenging. Despite the encroaching of low- and high-frequency approaches further and further within the mid-frequency range, and the development of specific hybrid methods, the question of how to deal with this range has not yet been put to rest. The author recently showed that it was possible to obtain frequency average and variance within the whole frequency range analytically, using the same integrals as those used to obtain the statistics of uncertain dynamic systems, if the probability density functions (pdf) are merely seen as filter or weighting function.

In principle, the resonances and modes of the system are required. Integrals need to be evaluated for each of the eigenvalues and then recombinined with the eigenvectors. The process of evaluating, storing, and recombinining the modes may however be too expensive and numerically unstable for dimensionally large systems, which may limit the applicability of the theory. It is highlighted here that it is possible to by-pass the modal analysis and work directly with solutions of the system. In the case of Gaussian averaging - the main focus here - only a few solutions may give the exact average around a central frequency up to several decimals. A stable Krylov scheme can be used to combine the solution vectors implicitly and provide the average very efficiently. Furthermore, a single rational Krylov scheme can provide an accurate approximation of the average in a whole frequency range of interest. This is done by working in a small reduced susbspaces spanned by solutions evaluated at only a few, relatively arbitrary, discrete frequencies. Here, the efficiency of the approach is illustrated on systems of different sizes and modal densities.

The theory is first presented in three steps. In section (2.), the statistical theory of propagation of uncertainty, applicable for any parameter variation such as a variation of the frequency of interest, is summarised. This straightforwardly results in the theory of frequency averaging. The idea of using rational approximations of the stochastic coefficients is introduced and the particular case of a real Gaussian function is briefly presented. In section (3.), it is explained that the rational approximations of the stochastic coefficients can be used to obtain a matrix, rather than scalar, expression of the stochastic coefficients. This removes the need for modal analysis and explicit storage and combination of modes. It is then stressed that the rational expressions can be evaluated implicitly through a stable rational Krylov method. Whole frequency ranges are considered in section (4.). It is proposed to combine rational Krylov subspaces at only a few interpolation frequencies, and to use the resulting reduced model to evaluate the averages in the whole frequency of interest.

The theory is then applied to two benchmark problems. The first one is used to demonstrate the efficient computability of the frequency averages, both by considering individual frequencies one at a time, and by using combined rational Krylov subspaces in a whole frequency range. The second one is used to illustrate the applicability of the proposed theory and techniques to work smoothly from the low- up to high-frequency ranges. The general theory, Krylov interpolations and their efficiency are discussed.

2. STEP A - FROM STATISTICAL AVERAGE TO FREQUENCY AVERAGE

In this section, the frequency average is defined and its analytical modal expression is obtained, from the theory of propagation of uncertainty in random dynamic systems.

2.1 Statistical average

Exact analytical mean and variance of transfer functions of dynamic systems with uncertainty have been presented recently in the context where the dynamic stiffness matrix, \( A(\omega) \), of the system is modified by a disturbance which is the product of a deterministic matrix, \( D \), by a scalar random variable, \( s \). The relation between input and output vectors, \( f \) and \( x(\omega, s) \) is then

\[
[A(\omega) - sD] x(\omega, s) = f
\]
and it can be shown that, for any value of \( s \), the disturbed output vector \( \mathbf{x}(\omega, s) \) can be expressed as a low rank update of its nominal value \( \mathbf{x}(\omega, 0) = \mathbf{A}(\omega)^{-1}\mathbf{f} \). For example, in the case of a rank-one disturbance, i.e. when the disturbance matrix, \( \mathbf{D} \), is rank-one and can be expressed as the outer product of two vectors such that, \( \mathbf{D} = \mathbf{d}_1\mathbf{d}_1^T \), the rank-one update is

\[
\mathbf{x}(\omega, s) = \mathbf{x}(\omega, 0) + s \frac{\mathbf{A}(\omega)^{-1}\mathbf{d}_1\mathbf{d}_1^T\mathbf{A}(\omega)^{-1}\mathbf{f}}{1 - s\mathbf{d}_1\mathbf{A}(\omega)^{-1}\mathbf{d}_1^T}. \tag{2}
\]

This standard Sherman-Morrison update can be demonstrated in different ways, including by using transfer function or modal approaches [1–4]. A critical advantage of the update is that it isolates the random variable and therefore permits a simpler and analytical evaluation of the means, variances, or covariances of all the transfer functions and response vectors. Notably, the mean of the transfer function, for any given pdf, \( p(s) \), of the random variable, \( s \), is

\[
\bar{\mathbf{x}}(\omega) = \int_{\mathcal{D}(s)} \mathbf{x}(\omega, s)p(s)ds = \mathbf{x}(\omega, 0) + \mathbf{A}(\omega)^{-1}\mathbf{d}_1\mathbf{d}_1^T\mathbf{A}(\omega)^{-1}\mathbf{f} \int_{\mathcal{D}(s)} s \frac{1}{1 - s\mathbf{d}_1\mathbf{A}(\omega)^{-1}\mathbf{d}_1^T} p(s) ds \tag{3}
\]

where \( \mathcal{D}(s) \) is the domain of \( s \). Thanks to such isolation of \( s \), the statistics can be obtained analytically from scalar integrals rather than by manipulating vector expressions and using approximations. Needed are scalar integrals or stochastic coefficients, i.e.

\[
e_1(\omega) = \int_{\mathcal{D}(s)} \frac{s_1(\omega)}{s_1(\omega) - s} p(s) ds \quad \nu_{11}(\omega_A, \omega_B) = \int_{\mathcal{D}(s)} \frac{s_1(\omega)}{s_1(\omega) - s} \bar{s}_1(\omega) - \bar{s}_1(\omega) \bar{s}_1(\omega) \frac{1}{s} p(s) ds \tag{4}
\]

where \( \bar{s} \) indicates the complex conjugate of \( s \), and where \( s_1(\omega) \) is the nominal function \((\mathbf{d}_1\mathbf{A}(\omega)^{-1}\mathbf{d}_1)^{-1}\). While the integrals aren’t necessarily trivial to evaluate, in general, it was however shown that in the notable case of a real or complex normal normal variable, \( s \), they have analytical forms that can be evaluated efficiently and reliably [2, 4, 5].

Multi-rank disturbances correspond to the case where the disturbance matrix can be expressed as the outer product of full-rank, rectangular matrices, \( \mathbf{D}_1 \) and \( \mathbf{D}_r \), i.e. such that \( \mathbf{D} = \mathbf{D}_1\mathbf{D}_r^T \). Their treatment differs from that for rank-one disturbances in two aspects. First, at each frequency, one needs to solve an eigenvalue problem of the dimension rank, say \( r \), of the disturbance. While this step can have a minimal cost and even be done analytically for lowerrank disturbances, the evaluation of a full modal analysis may be prohibitive both in terms of cost and in terms of storage, once \( r \) and the system dimension, \( N \), increase. Second, alternative pairs of scalar integrals, \( e_j(\omega) \), as in (4) need to be evaluated with functions \( s_j(\omega) \), \( j = 1, \ldots, r \) that are the finite eigenvalues of the matrix pencil \((\mathbf{A}(\omega), \mathbf{D})\), such that [4]

\[
[\mathbf{A}(\omega) - s_j(\omega)\mathbf{D}] \mathbf{\phi}_j(\omega) = \mathbf{0} \quad \mathbf{\phi}_j \neq \mathbf{0}. \tag{5}
\]

### 2.2 Frequency average

It was proposed by the author to use the analytical expressions of the mean and variance to deal with high frequency variability and modal density [6, 7]. The idea is to use the pdf’s as mere averaging or weighting functions of a variation \( \Delta \omega \) around any central frequency \( \omega \). Without having a probabilistic meaning necessarily attached to these functions, the expressions correspond to frequency, rather than statistical, averaging.

Specifically, in the case of a system with modal damping such that its dynamic stiffness matrix is \( \mathbf{K}(1 + \eta) - \omega^2\mathbf{M} \) for some mass and stiffness matrices \( \mathbf{M} \) and \( \mathbf{K} \) and damping ratio \( \eta \), the equations of motion for a variation \( \Delta \omega \) around any central frequency \( \omega \) are

\[
[\mathbf{K}(1 + \eta i) - (\omega + \Delta \omega)^2\mathbf{M}] \mathbf{x}(\omega, \Delta \omega) = \mathbf{f}. \tag{6}
\]
where the imaginary part of $s$ were presented in [2, 4]. In the case, used in further sections, where the mean, $\mu$, independent of the eigenvalues and that increasing the order, $K$, where $n$, these conditions. In the rest of the paper, for simplicity of presentation, the roots, $s_j$, eigenvalues, the matrix evaluation still applies as long as all the matrix eigenvalues satisfy as explained in section (3.). If there are conditions for the independence of the scalars on the case of real or complex Gaussian functions. Since the disturbance matrix $D$ can be full-rank, a modal analysis of the linearised form of equation (6) might be necessary to identify the alternative $s_j$ functions. This modal analysis is not necessary with the matrix or transfer function approach proposed here, starting from rational expansions of the stochastic coefficients.

### 2.3 Rational approximation of the averages

In the current work, rational approximations of exact expressions of the stochastic coefficients are used. For any eigenvalue, $s_j(\omega)$, of the matrix pencil $(A(\omega), D)$, they are such as, say,

$$e_j(\omega) = \lim_{K \to \infty} \sum_{k=0}^K c_k \left( \frac{s_j(\omega) - n_{k,K}}{s_j(\omega) - d_{k,K}} \right)^j$$

(8)

where $\sum_{k=0}^K$ denotes a sum. A critical point is that, the scalars, $c_k$, $n_{k,K}$, and $d_{k,K}$, are independent of the eigenvalues and that increasing the order, $K$, of a truncated sum leads to an approximation converging quickly independently of the value of these eigenvalues.

This leads to the matrix or transfer function, rather than modal, evaluation of the average, as explained in section (3.). If there are conditions for the independence of the scalars on the eigenvalues, the matrix evaluation still applies as long as all the matrix eigenvalues satisfy these conditions. In the rest of the paper, for simplicity of presentation, the roots, $n_{k,K}$, and poles, $d_{k,K}$, are supposed constant and, also, there is one term in the rational approximation for each exponent, i.e., $n_K = n_{0,K} = n_{1,K} = \ldots; d_K = d_{0,K} = \ldots; j_K = k$, so that

$$e_j(\omega) = \lim_{K \to \infty} \sum_{k=0}^K c_k \left( \frac{s_j(\omega) - n_K}{s_j(\omega) - d_K} \right)^k.$$ 

(9)

More general rational approximation cases nevertheless fall into the realm of the approach and theory here presented and simplifications may also occur. For example, in the illustrations of section (5.), two poles, 0 and $d_K$, and sets of solutions $A^{-1}v$ and $(A - d_K D)^{-1}v$ are apparently needed but since the first solution is premultiplied by $A$ and trivially simplify as $A(A^{-1}v) = v$, a single pole and Krylov subspace with solutions $(A - d_K D)^{-1}$ are necessary.

### 2.4 Particular case of a Gaussian function

In the particular case of a normal real variable, the pdf is a Gaussian function, so that

$$p(\Delta \omega) = p^{(s)}(\Delta \omega; \mu, \sigma) = \sqrt{\frac{1}{2\pi\sigma^2}} e^{-\frac{(\Delta \omega - \mu)^2}{2\sigma^2}}$$

(10)

for a mean $\mu$ and standard deviation $\sigma$. Analytical expressions of the stochastic coefficients were presented in [2, 4]. In the case, used in further sections, where the mean, $\mu$, is null and where the imaginary part of $s_j(\omega)$ is positive, the first stochastic coefficient is equal to

$$e_{1}^{(s)}(\omega) = \frac{s_1(\omega)}{\sqrt{2\pi\sigma^2}} I_1 \left( \frac{s_1(\omega)}{\sqrt{2}\sigma} \right)$$

(11)
where $I_1(b) = -i\pi w(b)$, the Faddeeva [8] function is $w(b) = e^{-b^2} \text{erfc}(-ib)$ [9, equation (7.1.4)], and the complementary error function is defined by $\text{erfc}(z) = \left(2/\sqrt{\pi}\right) \int_z^\infty e^{-t^2} dt$. While there is no closed form expression to evaluate the Faddeeva and error functions, these special functions have been extensively (and are still) studied over the last fifty years. There exist precise and efficient numerical methods to evaluate them reliably as shown in [4].

3. **STEP B - FROM MODAL TO SEMI-ANALYTICAL KRYLOV EXPRESSIONS**

In this section, a semi-analytical Krylov approach is presented. A matrix representation of the semi-analytical average is first introduced and it is then explained how it can be evaluated implicitly in a smaller subspace by using a rational Krylov projection method [7, 10–13].

3.1 **Matrix rational representation of the average**

In this section, it is explained how the frequency average, can be expressed in matrix form. It is assumed for simplicity of presentation that the mass matrix $M$ is full rank, that there is a full set of system eigenvectors, and that the $s_j(\omega)$ are all different. The eigenvalues and eigenvectors are used in the derivation but do not appear in the final expressions.

Grouping the eigenvectors in a matrix $\Phi = \begin{bmatrix} \phi_1 & \cdots & \phi_N \end{bmatrix}$, equation (5), i.e. $A(\omega)\Phi(\omega) = D\Phi(\omega)S(\omega)$, gives modal representations of the matrices $A$ and $D$, so that

$$A(\omega) = (D\Phi(\omega))S(\omega)(\Phi(\omega))^{-1} \quad \text{and} \quad D = (D\Phi(\omega))I(\Phi(\omega))^{-1} \quad (12)$$

$$S(\omega) = (D(\omega)\Phi(\omega))^{-1} A(\omega)(\Phi(\omega)) \quad \text{and} \quad I = (D(\omega)\Phi(\omega))^{-1} D(\omega)(\Phi(\omega)) \quad (13)$$

where $S(\omega) = \text{diag}_{j=1,...,N}(s_j(\omega))$ the diagonal matrix of eigenvalues and $I$ is the identity matrix. As usually done in structural dynamics or vibro-acoustics, matrix operations can now be expressed in modal form. In particular, the average of equation (3) can be written

$$\bar{x}(\omega) = \Phi(\omega)\text{diag}_{j=1,...,N}(e_j(\omega))\Phi^{-1}x(\omega,0). \quad (14)$$

Substituting the rational expression of equation (9) then gives

$$\bar{x}(\omega) = \lim_{K \to \infty} \left\{ \Phi(\omega)\text{diag}_{j=1,...,N} \left[ \sum_{k=0,...,K} c_{k,K} \left( \frac{s_j(\omega) - n_K}{s_j(\omega) - d_K} \right)^k \right] \Phi^{-1}x(\omega,0) \right\} \quad (15)$$

and, using again the modal expressions of the matrices, one finds the matrix expression

$$\bar{x}(\omega) = \lim_{K \to \infty} \left\{ \sum_{k=0,...,K} c_{k,K} \left( A(\omega) - d_KD \right)^{-1} \left( A(\omega) - n_KD \right)^k \right\} x(\omega,0). \quad (16)$$

This expression is exact and does not require evaluation of the modes. Approximations, $\bar{x}_K(\omega)$, of various orders can be obtained by truncating the sum at a finite maximum order $K$.

3.2 **Implicit rational Krylov evaluation of the average**

While truncating the sum in equation (16) provides rational approximations of the average, it may be preferable, from a numerical and efficiency points of view, to evaluate them implicitly. Considering a particular frequency parameter, $\omega = \sigma_j$, the first step consists in building a subspace that contains the vectors $\left( A(\sigma_j) - d_KD \right)^{-1}D$ for $k = 0, \ldots, K_j$. If these vectors are spanned by the columns of an otherwise arbitrary full rank matrix, $V_j$, the interpolation properties of Krylov projection methods [11, 14, 15] assure that the approximations...
obtained in a smaller dimensional subspace where \( P_j = W_j^T AV_j, Q_j = W_j^T DV_j \) are smaller, reduced, matrices and \( W_j \) is a left projection matrix, match exactly their counterpart, i.e.

\[
\tilde{x}_{K_j}(\sigma_j) = V_j \sum_{k=0,\ldots,K_j} c_{k,K_j} \left[ \left( P_j(\sigma_j) - d_k Q_j \right)^{-1} \left( P_j(\sigma_j) - n_k Q_j \right) \right]^k \left( P_j(\sigma_j)^{-1} W_j^T \right)^T f. \tag{17}
\]

Only a small number, \( K_j + 1 \), of solutions is necessary at each frequency. An advantage of the proposed formulation is that Krylov or rational Krylov, relatively standard, techniques can be used to generate the projection matrix \( V_j \), condense the matrices, and solve for the approximations. These approximations can notably be easily iteratively updated when \( K_j \) increases if the poles, \( d_k \), are independent of the approximation order, \( K_j \). While the expansions (9) and (17) are generally approximations for finite \( K_j \), there are situations in which the expressions are exact. Finally, one may only need a very small number of terms in the expansion for obtaining a given working precision of the average. This may be less than \( K_j = 10 \) for a precision of several decimals. Using the duality of the interpolation properties of rational Krylov interpolations, i.e. choosing an appropriate left projector, \( W_j \), one can further reduce the number of full solutions by half. Efficient rational expressions of the average exist for example in the case of real Gaussian, as well as Cauchy, averaging. Since the expansion (9) may be seen as one of the ”standard” numerical methods to evaluate the Gaussian stochastic coefficient functions in software packages such as Matlab or Mathematica, the Krylov scheme proposed here may be seen as a ”semi-analytical” method in this case.

4. STEP C - RATIONAL KRYLOV PROJECTION IN A FREQUENCY RANGE

The semi-analytical Krylov approach presented in the previous section may allow to evaluate the statistical or frequency average semi-analytically, i.e. up to a given precision, while requiring only a very small number of solutions at each frequency. When solving for the response of the system in a frequency range, the number of solutions may be further reduced by considering only a small number of (semi-analytical) interpolation frequencies, \( \sigma_1, \sigma_2, \ldots \). This is similar to one of the the approaches studied in [7, section 5.2] if only the approximations \( \tilde{x}_{K_j}(\sigma_j), j = 1, \ldots \) are retained in a projection subspace \( V \) to approximate the solution in the whole frequency range. An advantage of the implicit rational Krylov scheme is that the interpolation condition that assures that the expressions in equations (16) and (17) are equal is a condition of inclusion of spanned subspaces. It is therefore possible to combine the subspaces spanned by the columns of several matrices \( V_j \) into a single subspace spanned by the columns of a single matrix and still maintain the interpolation conditions at the interpolation frequencies, \( \sigma_j \). At any frequency, \( \omega_m \), the average is then approximated as (note the accent \( ' \) for the interpolated approximation)

\[
\tilde{x}_{L_m}(\omega_m) \approx \hat{x}_{L_m}(\omega_m) = V \sum_{l=0,\ldots,L_m} c_{l,L_m} \left[ \left( P(\omega_m) - d_{L_m} Q \right)^{-1} \left( P(\omega_m) - n_{L_m} Q \right) \right]^l P(\omega_m)^{-1} g. \tag{18}
\]

where \( L_m \) is the corresponding order, \( P = W^T AV, Q = W^T DV \) and \( g = W^T f \). The scalars \( c_{l,m} \), \( n_l \), and \( d_l \) have the same definition as before. Conceptual steps are presented in the pseudo-algorithm of figure (1). The term semi-interpolation refers to the fact that the semi-analytical average, \( \tilde{x}(\sigma_j) \), at the interpolation points is obtained up to a given precision, by truncating the sum of expression (9) at finite order \( K_j \); it is not per se an ”exact” evaluation but precision can be controlled and a large enough, but still reasonably small, \( K_j \) may allow to reach machine precision.

The matrices of step 4 can be evaluated by a standard rational Krylov approach. If a frequency \( \omega_m \) is equal to any interpolation frequency, \( \sigma_j \), and if \( L_m = K_j \) then there is interpolation, i.e. \( \tilde{x}_{L_m}(\omega_m) = \hat{x}_{L_m}(\omega_m) \). Some aspects deserve more study both from a theoretical
Algorithm semi-interpolation algorithm
1. Choose a number, J, of interpolation frequencies \( \sigma_j \) and corresponding orders \( K_j \)
2. for \( j \leftarrow 1 \) to \( J \)
3. Evaluate the pole \( d_{K_j} \)
4. Evaluate a full-rank matrix \( V_j \) whose columns’ span contains the vectors such as
   \[
   \left[ (A(\sigma_j) - d_{K_j} D)^{-1} D \right]^k A(\sigma_j)^{-1} f \text{ for } k = 0, \ldots, K_j
   \]
5. Evaluate a full-rank matrix \( V \) whose columns’ span contains all the columns of the \( V_j \)’s.
6. Choose a left projection matrix, \( W \), of the same dimensions as \( V \).
7. Choose one or more frequencies of interest, \( \omega_m \), and corresponding orders, \( L_m \)
8. for all \( \omega_m \)’s
9. Project the matrices and vector, \( P(\omega_m) = W^T A(\omega_m) V, Q = W^T D V, g = W^T f \)
10. Evaluate the root and pole, \( n_{L_m} \) and \( d_{L_m} \), and the coefficients \( c_{l,L_m} \) for \( l = 0, \ldots, L_m \)
11. Evaluate the semi-interpolated average \( \tilde{x}_{L_m} (\omega_m) \) of equation (18)

Figure 1. Algorithm for the semi-interpolated frequency average.

and practical point of view. So does notably the combination of the subspaces in step 5. Indeed, since the individual Krylov subspaces are generated at the interpolation frequencies with different matrices \( A(\sigma_j) \), the subspace spanned by the columns of \( V \) may not be in a single rational Krylov subspace if the linearisation such as that of equation (7) is not appropriately chosen. Constructing its basis sequentially or in parallel with Arnoldi iterations (i.e. with successive orthogonalisation) might then not be a trivial task since this may generate additional directions of a generalized Krylov subspaces (see [16]). Note that the important questions of working with parametric Krylov subspaces, similar to those here with different \( A(\sigma_j) \), and their theory has also been studied elsewhere, notably in [17, 18].

Important to stress, for applications, is that the matrix approach developed here can be also used for statistical averaging, i.e. in the context of propagation of multi-rank uncertainties.

5. ILLUSTRATIONS

Both semi-analytical and semi-interpolated frequency averages are illustrated on two benchmark problems. The focus of the first illustration is mainly on the comparison of the characteristics of the two types of evaluations. The goal of the second illustration is mostly to highlight the applicability of the frequency averaging to treat the low- to high-frequency ranges in a single analysis. It is shown, not only that the theory provides a framework to work smoothly through the frequency ranges, but also that the Krylov approach allows to do so by integrating information for the whole analysis within a single reduced model.

5.1 First benchmark - Semi-analytical averaging and its efficient Krylov interpolation

The first benchmark problem consists in the Matrix Market’s system BCSST11 of stiffness and mass matrices, \( K \) and \( M \) with dimension \( N/2 = 1473 \) [19, 20]. Damping is considered as viscous damping added to the system stiffness, so that the system equations are

\[
(K + i\omega C - \omega^2 M)x(\omega) = f \quad \text{where} \quad C = \eta K. \quad (19)
\]

The considered force acts on the first degree of freedom, so that the force vector is the first unit vector, \( f = e_1 \). Of interest is the response of the system at the same point, i.e. an output vector \( e = e_1 \), so that the corresponding transfer function is \( g(\omega) = e^T A(\omega)^{-1} f = e_1^T (K + i\omega C - \omega^2 M)^{-1} e_1 \). Differences with the benchmark used in [7] lie in the fact that smaller and viscous -rather than proportional- damping is used here, with \( \eta = 10^{-7} \). Note
also that the average is on \( \omega \) here rather than on the frequency parameter \( \omega^2 \) while the same theory and semi-analytical Krylov approaches are applicable in both cases.

The nominal and average transfer functions, \( g(\omega,0) \) and \( \tilde{g}(\omega) = c^T \tilde{x}(\omega) \), based on a rational approximation from [21], are presented in figure (2) for the particular case of a real Gaussian averaging function, \( p(\Delta \omega) = p(\Delta \omega; \mu = 0, \sigma = 50 \text{[rad/s]}) \), in the frequency range \([4690.4, 5000]\) \text{[rad/s]}. As should be expected, the average is a smoother function that follows the trend of the nominal transfer function. It is however neither constant, nor linear, as a function of \( \omega \). No numerical issues were apparent in the evaluation of the average.

![Figure 2: Comparison of nominal and frequency average transfer functions for the first benchmark with \( p(\Delta \omega) = p^G(\Delta \omega; \mu = 0, \sigma = 50) \) and \( K = 10 \).](image)

The average functions for various values of \( K \) are presented in figure (3). Differences are barely visible between the cases \( K = 2, 6, 10 \) and the magnitude of the relative difference between the \( K = 6 \) and 10 cases is only of the order of \( 5 \times 10^{-4} \) in the frequency range.

![Figure 3: Comparison and relative error of frequency average transfer functions for the first benchmark with \( p(\Delta \omega) = p^G(\Delta \omega; \mu = 0, \sigma = 50) \) and various numbers of solutions, \( K = 2, 6, 10 \). The errors for \( K = 2, 6 \) are relative to the average obtained with \( K = 10 \).](image)

As mentioned, the nominal solution \( x(\omega) = A(\omega)^{-1} f \) is not necessary for the evaluation of the average in this particular rational formulation, based on [21], since it only appears in the form \( A(\omega)x(\omega) = f \) which is trivially simplified. Properties of the Krylov subspaces are also put to use in order to further cut the number of solutions necessary in half. At any given frequency, the averages were therefore evaluated for \( K = 2, 6, \) or 10 with only 1, 3 or 5 full solutions (i.e. solutions in the original system space of dimension 1473). Remarkably, a single solution at each frequency provides the average up to the order of one percent precision.

Even more remarkable is the economy provided by the semi-interpolation algorithm when considering the whole frequency range. This is illustrated in figure (4) for two interpolation
points, i.e. for \( J = 2 \). Indeed, since the semi-analytical averages are only evaluated at these two interpolation points, only 2, 6 or 10 full solutions are used in the whole frequency range to obtain the interpolated solutions for \( K_j = 2, 6 \) or 10 respectively. At frequencies other than the interpolation frequencies, the averages are interpolated in a reduced subspace of dimension \( K_j \). As shown in figure (4), the interpolated average for \( K_j = 2 \) provides an approximation of the exact average that is again about one percent precise, at the main cost of only two full solutions in the space of dimension 1473 (one single solution at each of the two interpolation points). The interpolated average in the whole subspace can be extracted at negligible cost from the reduced subspace of dimension two. As shown, increasing the order, \( K_j \), of the approximation steadily improves the quality of the evaluated average. Evaluating only ten full solutions and, otherwise, working in a reduced subspace of dimension ten provides more than five correct decimals of the average in most of the frequency range of interest. As in the semi-analytical case, there was no apparent numerical issue in the analysis, when using a standard Arnoldi method for step 4 of the algorithm.

5.2 Second benchmark - Smooth and efficient frequency averaging from low- to high-frequency ranges

One now turns to the applicability of the averaging to deal with various frequency ranges of interest. No formal definition of low-, mid- or high-frequency ranges are necessary. A point of view might be that low-frequency corresponds to more deterministic analysis, i.e. small standard deviation, and that high-frequency is treated by a larger standard deviation so that individual modes and resonances are blurred into a global behaviour. The value of \( \sigma \) can simply be smoothly varied through the frequency ranges, so that the frequency averaging (including variance [2, 4–7]) applies smoothly through the whole frequency ranges. This is demonstrated here by considering frequency varying standard deviation, \( \sigma(\omega) \), and approximation order, \( K(\omega) \). These values at the interpolation points are such that, \( K_j = K(\omega = \sigma_j) \).

The considered benchmark is a circular system of springs and masses of dimension 565, such as that sometimes used to model bladed disks [4], where the normalised form of the

Figure 4: Comparison and relative error of interpolated frequency average transfer functions for the first benchmark with \( p(\Delta \omega) = p^{\sigma}(\Delta \omega; \mu = 0, \sigma = 50 \text{rad/s}) \) and various numbers of solutions, \( K_j = 2, 6, 10 \). The errors of the interpolated averages for \( K_j = 2, 6, 10 \) are relative to their corresponding (i.e. using the same \( K = K_j \)) semi-analytical average. The two interpolation points, marked by squares, are \( \sigma_1 = 4795.8 \text{rad/s}, \sigma_2 = 4899.0 \text{rad/s} \). The interpolated average is obtained in the whole range with only two (respectively \( K_j \)) solutions in the system space for \( K_j = 2 \) (respectively any \( K_j \)).

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system has a tridiagonal circulant stiffness matrix, $K$, with 2’s on its diagonal and −1’s on its two neighboring diagonals and at the locations (565, 1) and (1, 565). In order to somewhat complicate the spectrum, the 185th to 187-th diagonal elements are replaced here by 3.4 and the 260-th to 262-th by 2.2. The (normalized) mass matrix is the identity matrix, $M = I$. Damping is introduced as in equation (19), this time with $\eta = 0.1$. A unit input (force) is introduced at the first mass and the response at the 55-th mass is of interest, so that $f = e_1$ and $c = e_{55}$. The magnitude of the transfer function $g(\omega) = c^T A(\omega)^{-1} f$ is presented in figure (5) together with the computational objectives. While the whole frequency region is of interest, different objectives and levels of knowledge are usually desired in different regions. In this illustration, the response at lower frequency is desired precisely. This might be due to the fact the locations or magnitude of resonances and zeros are important system information. At higher frequency, where several resonances are highly damped, the exact location or identification of modes is not desired. Rather, the general level and average of the response is sufficient to know. The response in the transition region is desired with relative precision. Broad identification of modes or groups of modes is sufficient and neglecting modes that have small contribution to the response is acceptable or even desirable.

![Figure 5](image.png)

**Figure 5**: Magnitude of the nominal transfer function for the second benchmark and corresponding computational objectives. The frequency dependent orders $K(\omega)$ and standard deviations are presented in the plots to the right. Fourteen interpolation points are used and their corresponding order and standard deviation are marked by squares.

The values of the orders, $K$, and standard deviations, $\sigma$, are presented in the same figure, as well as their discrete values at fourteen selected interpolation points. The magnitude of the semi-analytical frequency average is presented in figure (6) and compared to the nominal transfer function. At low-frequency, the average essentially matches the poles, zeros, and transfer function values of the nominal function. The differences of response objectives are evident at mid-frequency, where some of the nominal poles are mostly absent from the averaged response and the other resonances and poles, although roughly identified, are smoothed out. At higher frequency, the individual poles and zeros become less and less visible. The choice of orders and standard deviation values through the frequency range is very flexible and can be adapted in real situations, notably, to satisfy particular engineering needs.

Running the semi-interpolation algorithm with the interpolation frequencies and their tuned dimensions and standard deviations presented in figure (5) gives an excellent approximation of the semi-analytic average. Again, it is hardly distinguishable visually. The relative errors, of the order of a few percents or less, are presented in figure (7). The interpolation frequencies are clearly identifiable. This approximation was obtained with a reduced model.
of dimension 63 while the number of resonances of the nominal system with real part smaller than 0.35rad Hz is in excess of hundred. The semi-interpolation algorithm provides a single reduced model with useful information tuned to the different frequency ranges.

![Figure 6: Comparison of the magnitude of the nominal and averaged transfer functions for the second benchmark. Zoomed frequency ranges are presented in the plots to the right.](image)

![Figure 7: Relative error of the second benchmark average using the parameters of figure (5).](image)

### 5.3 Illustration of the efficiency of the proposed approach

In this section, the efficiency of the proposed approach is compared on the second benchmark to that of estimating the average by direct sampling and averaging of the transfer function.

The analysis is run on a consumer desktop computer with the computational software Octave. While the solutions for the direct evaluations are evaluated in the original space of smaller dimension $N/2$, the solutions and other operations to build the Krylov subspace basis vectors, $V$, are evaluated in the linearised space of larger dimension $N$. Octave’s standard solver is used in all cases and timings are reported for illustrative purpose.

The average of the second benchmark is considered at $\omega = 0.075\text{rad Hz}$ for an averaging width $\sigma = 0.012$. For the proposed approach, the sum of the implicit rational Krylov expression (17) is truncated at various even maximum values of $k = K = 2, 4, 6, \ldots, 34$. The reference exact average, $\hat{g}(\omega = 0.075\text{rad Hz}) \approx -0.0248 - \imath 6.8571$ is obtained with $K = 40$.

A single Krylov subspace with shifted matrix $(A_\omega - d_kD)^{-1}D$ is necessary to obtain the rational approximation of [21]. Equal output and input vectors are chosen, $c = f = e_1$, and one can take advantage of the duality of the rational Krylov interpolation properties by using identical left- and right-projection matrices, $V = W$. The dimension of the Krylov subspace necessary to ensure interpolation is then reduced by half so that only $K/2 = 1, 2, 3, \ldots, 17$ full solutions are necessary to obtain the approximation (17) exactly up truncation $K$. 

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A simple Arnoldi scheme is used to construct the projection matrices such that $V^H V = I$ and $H = V^H (A(\omega) - d_k D)^{-1} DV$ is Hessenberg (i.e. all the terms below the line under its diagonal are zero). In this illustration, no advantage is taken of the facts that $A$ and $D$ have special form, as seen in equation (7), and that the solutions are all required with the same matrix. (The computational times could be significantly reduced by considering these facts.)

The resulting relative errors and computational times are presented in figure (8).

![Figure 8: Efficiency (precision-time) map comparing the proposed method to a direct evaluation of the average. Efficiency of the proposed method is presented for a total number of full solutions ranging from 1 to 17. Efficiency of the direct evaluation is presented for a total of 4 up to 24577 solutions in the range $[0, 0.15]$ rad Hz and about two thirds of that in the interval $[0.025, 0.125]$ rad Hz. The corresponding frequency steps range from $0.05$ to $6.1 \times 10^{-6}$ rad Hz.]

These performances are compared to those of the direct evaluation obtained in the following way: values of the nominal transfer functions are evaluated at regular frequency intervals in the frequency region, $[0, 1.5]$ rad Hz and then simply scaled by the corresponding values of the Gaussian averaging function and integrated by a trapezoidal rule.

For the same precision (and the particular implementations chosen), the proposed approach is more than two orders of magnitude faster. Furthermore, the precision obtained through direct evaluation is limited by the fact that the interval, and thus the averaging Gaussian, is truncated at 0 and 1.5 rad Hz. Using a shorter interval $[0.25, 1.25]$ rad Hz only slightly reduces the cost of the direct evaluation (it is still orders of magnitude larger than that of the proposed approach) while this significantly reduces its maximum attainable precision.

For the particular benchmarks presented here, the efficiency of the modal evaluation of the exact statistics lies somewhere in between the proposed matrix approach and the direct evaluation. The proposed matrix approach is expected to be even more advantageous for larger dimension systems as it has no need for storage (or evaluation) of the system modes.

6. CONCLUSIONS AND FUTURE DIRECTIONS

It was recalled in this paper that the theory of statistics of uncertain dynamic systems can also be used to evaluate frequency average and frequency variance of dynamic systems. Depending on the probability density function or averaging function, possibly difficult scalar integrals are necessary in the expressions of the averages and variances. It was stressed that if these integrals have rational expressions of the system eigenvalues, then it may be possible to obtain the averages of the transfer functions of the system, based on matrix operations that do not require modal analysis. This is for example the case for real Gaussian pdf’s.
Furthermore, using Krylov methods allows to evaluate the averages in a stable manner at each frequency of interest. Combining the Krylov subspaces at only a few frequencies gives additional advantages: a) there is interpolation of the averages at these *interpolation frequencies* and b) excellent approximations of the statistical or frequency averages in a whole frequency range may be obtained through operations in a dimensionally small subspace. The generation of this subspace necessitates only a small number of solutions within the larger original system space. It was demonstrated that as little as one solution at a given frequency of interest, or a handful of solutions -as little as two- for a whole frequency range may be sufficient to evaluate the average accurately.

It was stressed that the standard deviation of the averaging functions can be tuned at different frequencies in order to reach different computational objectives such as deterministic identification of resonances and zeros at low frequency and energy averaging at high frequency. The whole frequency range can therefore be treated in a single computational run. Furthermore, a single Krylov reduced model may contain all useful information for such multi-frequency range analysis. This capacity to cross the low- to high-frequency ranges with a single framework and analysis will be further discussed elsewhere by the author, together with a physical interpretation of the frequency averaging and variance. Particular useful in the context of this suggested treatment of the whole frequency ranges in a single framework through frequency average and variance, is research on a simplified matrix expression of the variance to complement the simplified matrix expression of the average presented here. While the expression of the frequency variance is computable based on its modal expression, as was demonstrated in [6, 7], a specific simplified matrix expression would likely allow practical applicability at much higher frequency, without requiring the evaluation and storage of the modes, and would permit significant additional theoretical developments.

While the theory was presented formally in the context of systems represented by matrices and vectors, it can be similarly derived and directly applied in the context of continuous systems described by linear operators and functions rather than by matrices and vectors. Such particular cases occur for example, in the same engineering dynamics context, if the responses and transfer functions of the system are expressed in alternative form, such as in the form of a sum of wave components at each frequency. The matrix and Krylov approaches can also be applied in the context of propagation of uncertainty in dynamic systems as long as the dynamic stiffness matrix or operator is modified by a single random variable times a rank-one, full-rank, or generally multi-rank matrix. The expressions of the statistical or frequency average (or variance) can also be applied for any pdf or averaging functions as long as their stochastic coefficients can be evaluated, for example, for uniform or Cauchy distributions used in the dynamics or physics communities, as in [22, 23].

While the theory was presented for the case of a system having a full set of eigenvectors, this condition is not necessary and the matrix expressions are still valid even if it is not satisfied. Linearisability of the system in terms of the frequency or random parameter in a form as in equation (7) is necessary, although this requirement may be seen as somewhat theoretical as a particular linear form is not really necessary and it can be seen as a mere step in the derivation. Any polynomial dependence on the parameter assures such linearisability.

Regarding the semi-interpolation of the average presented here, future developments include the derivation of error bounds, both at and between the interpolation frequencies, as well as detailed study of the generation, condensation, and solution of the Krylov projection matrices and reduced system. For example, working in the larger linearised (or state-space) space is not necessary and, reusing information for each solution in the generation of $V$ would be very beneficial when possible. More theoretically, it will be very useful both for practical and theoretical matters to study optimal choices of rational expressions such as that of equation (8) and optimal choice of the semi-interpolation points used in equation (18).
Finally, future work includes further study of averaging along several parameters. A better understanding of the common structures that involve several matrices, such as that of generalized Krylov subspaces [16], should be very beneficial in this regard.

REFERENCES


[8] Vera N. Faddeyeva and N. M. Terent’ev. *Tables of values of the function w(z) = e^{−z^2} \left(1 + \frac{2i}{\sqrt{\pi}} \int_0^z e^{−t^2} dt\right) for complex argument*. Pergamon Press, New York, Oxford, London, Paris, 1961. Traducted from original Russian, Gostekhizdat, Moscow, 1954.


