ALLOWING FOR NON-PROPORTIONALITY IN MULTIAXIAL SPECTRAL FATIGUE ANALYSIS.

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Keywords: Multiaxial Fatigue, Non-Proportional Loading, Spectral Fatigue

ABSTRACT
Spectral methods of life estimation, using Power Spectral Densities (PSD) are in regular use in the case of uniaxial fatigue. They are often linked to Finite Element Analysis (FEA). Recently their use has been extended to the field of multiaxial fatigue, again commonly linked with FEA. This paper examines some of the techniques, particularly ones linked to the non-proportionality which is inherent in such loading. The aim is to identify strategic issues in choosing a design method.

Fundamental reasons for the difficulties caused by non-proportionality lie in the complexities of the crack initiation process caused by fatigue, even when the loading history is simple and uniaxial. This has been, and still is, the subject of intensive research, Abuzaid, et al. \cite{1} being a typical modern reference. The details of material behavior needed to apply such considerations to the design of complete structures require computations too demanding for regular use, though. Instead, a variety of simplified models, such as strain-life analysis and fracture mechanics, are employed. Loading histories used by modern methods also impose simplifications. Probabilistic data reduction techniques like rainflow counting and frequency domain analysis based on power spectral densities (PSDs) are in regular use.

But, methods for calculating appropriate “equivalent” stress parameters like principal stress planes are not yet fully accepted. This paper reviews recent work in this field and then proposes new methods suitable for rapid fatigue design in the frequency domain.

1 INTRODUCTION
Many engineering components experience time varying stresses in more than one direction. The increasing use of FEA to evaluate the fatigue resistance of engineering systems has emphasized the importance of allowing for this. Standard FEA stress solvers calculate the stress tensor with multiple stress components at each output location in the model; but usually only one of these is considered when estimating the life of the component, leading to conventional uniaxial analysis. If all the other components are zero this is valid, but this condition is rarely satisfied for all elements in a large FEA model. If any of the other components in the stress matrix is non-zero its influence on component life should be considered. This leads to the topic of multiaxial fatigue analysis.

Gough \cite{2} tested steels using combinations of bending and torsion, and developed expressions which are still in use. Grubisek \cite{3}, reported biaxial tests in 1976, and offered “Test results concerning the behavior of carbon steel under constant and changing directions of principal stresses, including the effect of mean stress, phase difference and stress
concentration”. This recognizes principal stresses as important variables, and goes further by distinguishing between ones which are fixed in direction and ones with variable orientation. Modern analysis has made substantial progress for the first condition but the latter is proving difficult. The description “non-proportional” is now in common use to describe this occurrence.

A very simple non-proportional loading occurs in a cylindrical component carrying torsion, \( M_T \), about the \( x \) axis and bending, \( F_B \), about the \( y \) one. Conventional calculations then give the normal and shearing stresses on, say, the \( x \) face at any point. At any instant the orientation of the principal stresses will depend on the ratio of normal stress to shearing stress on that face. This in turn will depend on the ratio of the applied loads, \( M_T/F_B \), at that instant. If in the time domain both torsion and bending follow constant amplitude sinusoidal cycles of the same frequency, and both are zero at time zero this ratio will not vary in the time domain. Introducing a phase shift by, for instance, starting with the torsion input at maximum value when bending is zero, causes the ratio to vary with time, and so causes rotation of the principal stresses as time elapses. Experiments have shown that this change often shortens fatigue life.

Many service loadings are random in the time domain and are therefore intrinsically non-proportional. Spectral methods have been useful for these in the un-axial fatigue field and may be useful in the multiaxial one. The excitation, specified as a PSD then has components at many frequencies, and analysis must consider the influence of phase, which causes non-proportionality.

1.1 Searching for techniques; the approach used

Life estimation under realistic fatigue loading is commercially important, and literature about the subject is extensive. To manage this the sections below consider in turn:

- Section 2: Typical testing procedures
- Section 3: Some reported effects of principal stress orientation
- Section 4: Methods of analysis to allow for these effects.
- Section 5: Extension of existing methods for determining critical planes for multi input frequency domain loading

By adopting this structure it is possible to:

- Eliminate reports which do not use modern concepts for life estimation
- Establish what features of non-proportional loading influence fatigue life before considering how to allow for their effect.
- Propose methods to evaluate multi input non-proportional loading that can be used to evaluate fatigue life

2 TYPICAL TESTING PROCEDURES

Many reports of multi-axial testing have included non-proportional conditions. Some of these have been carried forward from uniaxial fatigue testing. Typically these tests do not simply measure life to total collapse, but aim to detect crack initiation and then monitor propagation. Life estimation methods also use the same division. Following again the experience of uniaxial research, especially the period 1960 to 1980, some factors which are often allowed for are:

- Non-linearity in stress/strain relationships. A three-parameter expression (e.g. Ramberg-Osgood) is common.
- Changes in these relationships under early cyclic loading; work-hardening and work-softening. These are especially important if short lives are being considered (low cycle fatigue or LCF).
- Modification of uniaxial crack propagation expressions (e.g. Paris-Erdogan) to accommodate more than one stress.
- The need to predict crack growth directions. There are more cases of Mode II growth, as compared with the dominance of Mode I in the uniaxial case.
- Mixed-mode propagation, requiring complex expressions for crack-tip stress intensity factors, $K$.
- Special treatment of cracks shorter than about 2 mm; crack closure and retardation.

Requirements for valid tests include:
- A precise description of specimen geometry
- Minimum crack length detected, and ranges of length given special treatment.
- A well-controlled criterion for final life.

Some common practices are:
- **Specimen geometry**. Two forms of specimen are dominant for multi axial testing. These are:

  ![Specimen Images](image)

  Figure 1. Most common forms of multi axial test specimen

  - (a) The thin-walled tube
  - (b) the cruciform

- **Crack length**. Current practice is to divide life into:
  - A *crack initiation period*: From a polished surface to a ‘crack’ with a ruling dimension of about 30$\mu$m
  - A *short crack propagation period*: From 30$\mu$m to 2 mm ruling dimension.
  - A *main crack propagation period*: From 2 mm to termination.

- **Termination of test**. A stated percentage loss of stiffness is a common criterion. Percentage drop in pressure is used when internal pressure is a load.

### 2.1 The concepts of “critical plane” and “equivalent stress”

Choosing parameters so that valid comparisons can be made is more difficult in the multiaxial case than it is in the uniaxial one. This begins even when test conditions are chosen. It is desirable to have combinations of, for example, normal stress and shearing stress which allow coherent comparisons to be made. Sometimes the parameter chosen will also feature in the analysis being investigated. The Tresca and von Mises expressions for static yield are often used. One group of life estimation proposals are classified as “critical plane” ones. These depend on identifying the plane on which damage will be accumulating most quickly. This needs some link between damage and the stress or strain history. Often this is based on equivalence, an “equivalent” stress or strain being a uniaxial value which will cause the same damage as the specified multiaxial combination. In many proposals this extends to allowing for non-proportionality. Any numerical information about the conditions, including the orientation of principal stresses, may be introduced at this stage.

### 3 SOME REPORTED EFFECTS OF PRINCIPAL STRESS ORIENTATION

Evidence about the effect of non-proportionality is conflicting, but compiling a comprehensive summary is outside the scope of this paper. A more limited but useful target is to ask:
Does non-proportionality affect life?
If it does, can the parameters causing the effect be identified?
If the parameters can be identified, can their contribution be quantified to provide life estimates?

Considering all the parameters which have been used in multi-axial theories is also unnecessary. One of the main aims of this paper is to identify appropriate procedures when frequency based (spectral) methods of stress analysis are used. Controlled rotation of principal stress direction is likely to provide quantitative information about the non-proportionality which is always present when these are needed. Reports in this field regularly use either strain path analysis or the concept of equivalence described in Section 2.1 above. Attention was therefore focused on these.

Using these principles, and taking only reports which conform to Section 2, directed our attention towards the following references.

3.1 Fatemi and Socie, [4] introduction of strain paths

Using thin-walled tubes of SAE 1045 steel, these tests used combinations of axial and torsional constant amplitude sinusoidal strain. The measure of strain amplitude for each test was chosen using the “von Mises equivalence” criterion of:

\[
\frac{\Delta \varepsilon}{2} = \sqrt{\left(\frac{\Delta \varepsilon}{2}\right)^2 + \frac{1}{3}\left(\frac{\Delta \gamma}{2}\right)^2}
\]

(1)

where \(\Delta \varepsilon\) is the amplitude of axial strain, \(\Delta \gamma\) is the amplitude of torsional strain and \(\Delta \varepsilon\) is the amplitude of von Mises equivalent strain. Orientation of the critical plane for the in-phase tests was varied by choosing different values of \(\Delta \gamma/\Delta \varepsilon = \lambda\). Failure was defined as a 10% drop in axial load (or in torque for pure torsion tests). Results are listed in Table 1.

<table>
<thead>
<tr>
<th>(\lambda)</th>
<th>(\Delta \varepsilon)%</th>
<th>(\Delta \gamma)%</th>
<th>(N_{\text{in-phase}})</th>
<th>(N_{\text{out-of-phase}})</th>
<th>(N_{\text{out-of-phase}}/N_{\text{in-phase}})</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.5</td>
<td>0.41</td>
<td>0.2</td>
<td>11,777</td>
<td>5,260</td>
<td>0.45</td>
</tr>
</tbody>
</table>
| 0.5 | 0.21 | 0.11 | 92,000
115,462 | 58,525 | 0.56 |
| 1 | 0.37 | 0.37 | 10,377
11,611 | 5,119 | 0.47 |
| 1 | 0.19 | 0.19 | 103,000
123,544 | 49,143
64,642 | 0.50 |
| 1 | 0.14 | 0.14 | 393,634
595,613 | 1,391,710 | 2.81 |
| 2 | 0.26 | 0.52 | 20,031 | 5,262 | 0.26 |
| 2 | 0.15 | 0.28 | 98,779
101,000 | 34,718
34,194 | 0.37 |
| 2 | 0.10 | 0.19 | 545,840
613,554 | 613,554 | 1.12 |
| 25 | 0.26 | 0.52 | 20,031 | 4,350 | 0.22 |
| 25 | 0.15 | 0.28 | 98,789
101,000 | 18,325
18,325 | 0.22 |
| 25 | 0.10 | 0.19 | 545,840
91,948 | 91,948 | 0.17 |

Table 1. The influence of principal stress orientation [4]

Although there are only a few tests the ratio \(N_{\text{out-of-phase}}/N_{\text{in-phase}}\) in column 6 is clearly less than one at lives shorter than about 100,000. To explain this effect the authors suggest a modification of the analysis by Brown and Miller [5], “to account for the additional cyclic hardening in out-of-phase loading”, and take the discussion to ‘strain paths’. When the
loading is out-of-phase strain paths on significant planes are complex. For the 90° case used in this report they are ellipses with an axis ratio which depends on the value of $\lambda$. These patterns may be used in analysis, a topic which has had much attention.

### 3.2 Itoh [6], more on the strain path effect

This report is an example of more extensive investigation of strain path effects. Hollow cylindrical specimens were loaded using combinations of axial load and torsion. Materials used were Type 304 steel and 6061-T6 aluminum alloy. Failure was defined as a 5% drop in axial stress from its cyclically stable value. Investigation of LCF was the declared intention. The 14 strain paths used are shown in Figure 2.

![Figure 2. The strain paths used by Itoh [6]](image)

The authors suggest dealing with non-proportionality by using two factors, $\alpha$ and $f_{NP}$, to calculate a modified effective strain range, $\Delta \varepsilon_{NP}$, by applying:

$$\Delta \varepsilon_{NP} = (1 + \alpha f_{NP}) \Delta \varepsilon$$

(2)

This is discussed in Section 4. Applying it to the 14 strain paths gives Table 2

<table>
<thead>
<tr>
<th>Case</th>
<th>$f_{NP}$</th>
<th>$f_{NP}$</th>
<th>$f_{NP}$</th>
<th>$f_{NP}$</th>
<th>$f_{NP}$</th>
<th>$f_{NP}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0.34</td>
<td>0.34</td>
<td>0.39</td>
<td>0.39</td>
<td>0</td>
</tr>
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<td>0.34</td>
<td></td>
<td></td>
<td></td>
<td>0.10</td>
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<tr>
<td>2</td>
<td>0.34</td>
<td>0.39</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
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<td>0.39</td>
<td>0.39</td>
<td></td>
<td></td>
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<td></td>
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<td></td>
<td></td>
<td></td>
<td>0.10</td>
</tr>
</tbody>
</table>

Table 2. Values of the non-proportionality factor [6].

Using this on the test results gives Figure 3. The effect of non-proportionality (white circles) has been properly estimated.

![Figure 3. Test results plotted against Itoh’s proposal [6]](image)
In the same project similar specimens of aluminum 6061-T6 were not sensitive to path shape.

### 3.3 Skibiki [7], investigation of some specific phase values

This reports tests using axial force and torsion on solid cylindrical specimens. Principal stress orientations of zero, 7.5°, 15°, and 22.5° were used. Materials were X5CrNi18-10 and AW 6063 steels. Figure 4 shows that in X5CrNi18-10 phase shifts caused life changes, greater shifts causing greater changes. In Figure 5 the black or grey blocks are lives without phase shift and the white ones are tests with phase shift, higher test numbers being greater phase shifts. This confirms the effect in Figure 4 for X5CrNi18-10, but AW 6063 shows no changes.

![Figure 4. The effect of principal stress angle on steel X5CrNi18-10[7]](image)

![Figure 5. Test results for two materials: (a) Material AW 6063 (b) Material X5CrNi18-10](image)

These tests and the ones in [6] show that changes in principal stress orientation cause damage changes in some materials. When it does occur, greater angle deviation usually causes more damage.

### 3.4 Nieslony and Bohm [8], non-proportionality with excitation in spectral terms.

This reports tests using excitation specified in spectral terms. The specimen was a cruciform plate with transverse holes Figure 6. Loading was the random history shown in time and frequency form in Figure 6b. This was applied in two orthogonal directions as $F_{1-3}$ and $F_{2-4}$. Two test series were run, a proportional one with $F_{1-3} = F_{2-4}$ and a non-proportional one with $F_{1-3} = -F_{2-4}$ (a phase difference of 180°)
The most notable feature of the tests was that the pattern of cracks formed by the two loading histories was substantially different (compare Figure 6(a) with Figure 7). FEA analysis gave different fatigue damage maps. Any examination of the effect of phase shift must take account of this possibility. Because of this, interpretation of the results takes more space than can be allowed here. The non-proportional loading did cause more damage, though.

### 3.5 Nieslony et al. [9], using uniaxial spectral formulae.

This reports tests with about twenty loading patterns all specified by PSDs. Results were compared with estimates made using modifications of methods accepted in the uniaxial case. These use empirical links between the properties of the PSD and life estimation methods such as rainflow counting. The proposed analysis depended on estimating an “equivalent PSD”, discussed below.

More than one hundred tests were compared with predictions. The specimen used was unusual. It was a thin-walled tube with a transverse hole. Failures originated inside the hole, usually being a crack about 1 mm long. FEA was used to search all elements in the critical region, using the Smith-Watson-Topper (SWT) expression for the strain/life relationship. The Rayleigh formula, the Wirsching-Light method, the Dirlik [10] formula and the Benessciuti-Tovo approach were used. The results showed that the “equivalent PSD” performed exactly as a uniaxial PSD would. The Dirlik and the Benessciuti-Tovo methods gave good estimates. A sample result is given in Figure 8.
Figure 8. Sample of life estimates from reference [9].

3.6 General comments on Section 3

Some guidance can be derived even from this small selection of references. The main findings are:

- Principal stress orientation does affect life in some materials
- The effect of principal stress orientation is more pronounced for LCF
- When using FEA, algorithms quick enough to scan whole models are desirable, even if only used as preliminaries.

4 METHODS OF ANALYSIS TO ALLOW FOR THESE EFFECTS

The reports in Section 3 show that both strain path analysis and the concept of “equivalence” detect non-proportionality and give some quantitative information. This section examines the available computation methods in more detail to assess the suitability of each as a design aid.

4.1 Strain path: Itoh [6], Test of a specific strain path approach

Early in the document it is stated:

“The change of the principal stress/strain direction due to non-proportional straining increases the interaction between slip systems, which is the cause of the additional hardening.”

Clearly a physical basis is being sought for a model. This is followed by:

“Two-surface plasticity model based on kinematic hardening rule the stress and strain state is expressed in terms of deviatoric vector planes. The definition of the axial torsional subspace follows as an Ilyushin’s five-dimensional deviatoric vector subspace and a stress vector”

The basis of this model is described more fully in Itoh et al. [11]. Applying it enables the construction of strain path diagrams such as Figure 9.
Parameters derived from these strain paths are then used to compile tables like Table 1 in section 3.2. This requires a factor $\alpha$ and a value for $f_{NP}$ in eq (2). Derivation of $\alpha$ and $f_{NP}$ is supported by:

“$\alpha$ is the material constant which discriminates the material dependency of additional hardening, and $f_{NP}$ the non-proportional factor which expresses the severity of non-proportional loading. The value of $\alpha$ is defined as the ratio of stress amplitude under 90 degree out-of-phase loading (circular strain path in $\gamma$ plot)”

An expression for $f_{NP}$ is given in the paper. It is fairly complex and requires an integration. It would be a key factor when using this approach with FEA. Complete scans of large models would take an unreasonable time.

4.2 Strain path: Proposal by Skibiki [7]

A version of the critical plane approach is proposed. It is often assumed that once the critical plane has been established the rate of damage accumulation will be controlled by some function of the normal and shearing stresses on that plane. In this case shearing stress is taken as the main criterion, and it is assumed that the parameter should be should the actual shearing stress, $\tau_a$, modified by the amplitude of normal stress, $\sigma_a$, and its mean $\sigma_m$, so that

$$
\tau_{eq(a)} = (\tau a + c_1 \sigma_a + c_2 \sigma_m)
$$

This is a common device, but in this case the paper goes on to propose that

$$
c_1 = 1.9 \left( \frac{t_{-\ell}}{b_{-\ell}} \right), \quad c_2 = 0.5 \left( \frac{b_{-\ell}}{R_m} \right)
$$

where $t_{\ell}$=fatigue limit in torsion, $b_{\ell}$=fatigue limit in bending and $R_m$=tensile strength.

The paper goes further, and introduces an equivalent shearing stress which allows for non-proportionality. Denoting this by $H$, the equation suggested is

$$
\tau_{eq(a)}^{NP} = \tau_{eq(a)} \left( 1 + \frac{t_{-\ell}}{b_{-\ell}} H^3 \right) \leq t_{-\ell}
$$

In deriving $H$ two assumptions are proposed:

(a) that it is directly proportional to modules of stresses acting beyond the critical plane.

(b) that it depends on their angular distance in such a way that the sectors acting in a larger distance in relation to the critical plane increase the loading non-proportionality degree more than the vectors of the same module acting within a smaller angular distance.
It is then argued that “assumption (a) was accounted for through application of a filling factor defined as the ratio of the loading path field of reduced stress to the field of a circle circumscribed about the loading path”. This is accompanied by a diagram Figure 10.

Figure 10 Loading path needed for Skibiki analysis [7].

Assumption (b) is suggested as a basis for estimating a weighting factor to be applied when allowing for non-proportionality, but this is not pursued.

As in Section 4.1 one of the operations has a computationally intensive item, in this case creation and analysis of a strain path.

4.3 Equivalence: Proposal by Nieslony and Bohm [8]

The objective of this investigation was to test the relevance of some uniaxial spectral techniques for multiaxial analysis. These use single parameter expressions and an equivalent PSD is required. The starting point is von Mises, but it is pointed out that this is only valid if the inclination of the S-N curves for tension-compression, \( w_f \) and \( w_t \), the fatigue exponents in push pull and torsion are equal and the fatigue limits in push pull and torsion are related by \( f_{-1N} = \sqrt{3} t_{-1N} \). To avoid this an expression for the equivalent PSD, \( G_{eq}(f) \) is formulated directly in the frequency domain

\[
G_{eq}(f) = g \left[ G_{t_{1N}}(f), G_{c_{1N}}(f), P \right] \tag{6}
\]

where \( G_{t_{1N}}(f) \) and \( G_{c_{1N}}(f) \) are determined using the Trace functions of matrices specifying stresses and \( P \) is the vector of material constants. Much earlier in the paper is the statement:

“the basis of the new approach consists of introducing the concept of damaging stress, which can be expressed for any loading in the complex form as

\[
\sigma(t) = \sigma(t) + ik \tau(t) \tag{7}
\]

where \( \sigma(t) \) and \( \tau(t) \) are the final realizations of a random process of normal and shear stresses independent from each other and \( k \) is a real coefficient defined as the ratio of the fatigue limits in fully-reversed push-pull \( f_{-1} \) and fully reversed torsion \( t_{-1} \).

The variance of the process is thus computed as

\[
s_{d}^{2} = s_{\sigma}^{2} + k^{2}s_{\tau}^{2} \tag{8}
\]

Much of the paper is devoted to the question of correlation between stress tensors on different planes, and the off diagonal elements which control this, which may have real or imaginary components. At one point an equation is used which needs two constant factors, \( a_{N} \) and \( b_{N} \).

\( a_{N} \) is put equal to \((\sqrt{6}.f_{-1N}/(2.t_{-1N}))\), but the expression for \( b_{N} \) needs integration unless it is assumed that \( \tau_{on}(t) \) and \( \sigma_{on}(t) \) are correlated, when it becomes -sqrt(2).\( a_{N}+3 \). Steps were taken in the experimental program though to avoid correlation.

These methods need an expression linking life with the amplitudes of stress or strain. If data is available the Manson/Morrow equation is to be preferred, and was used.
4.3.1 An example of the use of the equivalent stress approach in design [15]

The approach proposed in the Nieslony report, in particular the use of the *Trace* of the stress function, has been used in an industrial design environment. Bonte et al. [15] added a MATLAB routine to conventional FEA software using customer correlation data as described in the flow chart in Figure 11. This successfully predicted the location of a crack which occurred on a later full-scale test of a 12-wheeled truck.

![Flow chart for life estimation based on equivalent stress](image)

Figure 11. Flow chart for life estimation based on equivalent stress [15]

4.4 General comments on Section 4

Strain path methods
- Both methods need quite complex strain path analysis
- The criteria used for damage seem to be still speculative
- Links with spectral analysis are weak
- Computation time may be too high for general scanning

Equivalence methods
- Although the assumption proposed by Preumont, et al. [12] is empirical, experiments show that it does give accurate predictions in many circumstances where other methods fail.
- Input loads specified by PSD are common in modern design, and fatigue life estimation directly from a PSD is in regular use for uni-axial loading. Papers like [9] show that the Preumont assumption may extend this technique to the multiaxial case.
- The assumption has been verified by multiple tests on small specimens [9], using close control of the location and early propagation of cracks
Other tests,[8], show that when several alternative locations in the test specimen have similar damage potential phase differences between inputs may cause life changes which are not predicted by this method.

In testing a method of analysis, close observation of initial crack location is essential. Comprehensive scans of quite large areas are desirable when using FEA.

5 METHODS FOR DETERMINING CRITICAL PLANES FOR MULTI INPUT FREQUENCY DOMAIN LOADING

This section will develop relationships for multi input random loading that are extended in order to evaluate a general complex stress tensor to find maximum principal stresses and or maximum shear stress critical planes.

5.1 Complex stress in frequency domain

Consider a general frequency domain state of stress \( S(f) \). There are several important points to consider:

- A stress component at a given frequency rotates from a positive maximum value to a negative minimum value with some magnitude and complex phase relationship to the input.
- \( S(f) \) evaluated at \( f \neq 0 \) has a zero mean, i.e. a mean stress value is actually a zero frequency component
- Stress components (e.g. \( S_{xx}(f), S_{yy}(f) \)) can be related to one another by a complex factor \( a + ib \) where the components are
  - \( b = 0, a > 0 \); in phase
  - \( b = 0, a < 0 \); 180° out of phase
  - \( a = 0, b > 0 \); 90° out of phase
  - \( a \neq 0, b \neq 0 \); arbitrary phase

5.2 Stress tensor evaluated at an arbitrary orientation

For the current topic of multi axial fatigue, the failure models require an understanding of the maximum principal and or the maximum shear stresses; to do this one needs to evaluate the stress tensor. For real valued stresses, the principal stresses and directions can be determined based on eigen values of the stress tensor \( S \). Once principal stresses are known, it is a straightforward matrix math problem to determine the max shear stresses and their directions based on coordinate transformations.

One can also determine the stresses on a surface at an arbitrary orientation by matrix math. Consider the 3x3 symmetric stress tensor \( S \), where the components relative to a local x, y, z coordinate system are

\[
S = \begin{bmatrix}
S_{xx} & S_{xy} & S_{xz} \\
S_{yx} & S_{yy} & S_{yz} \\
S_{zx} & S_{zy} & S_{zz}
\end{bmatrix}
\]  

where the stresses normal to the local coordinate system are \( S_{xx}, S_{yy}, S_{zz} \), and the shear stresses are \( S_{xy} = S_{yx}, S_{xz} = S_{zx}, S_{yz} = S_{zy} \). Define the arbitrary orientation relative to the local coordinates by a direction cosine vector (DCV) \( \psi = (l, m, n)^T \) normal to the surface plane of interest, where \( l^2 + m^2 + n^2 = 1 \), i.e. the DCV is a unit vector.

The state of stress projected on the plane defined by this DCV \( \psi \) is a vector

\[
\bar{S}_p = S\psi
\]
The stress normal to the plane is a scalar found using the dot product
\[ \bar{S}_N = \psi \bar{s}_p = \psi' \bar{s}_p = \psi' S \psi \]  
(11)
and the shear stress parallel to the plane is found using the cross product
\[ \bar{s}_r = \psi \times \bar{s}_p, \]  
(12)
where the shear stress is a vector with magnitude and DCV \( \psi \)
\[ \bar{S}_r = |\bar{s}_r|, \psi = \frac{\bar{s}_r}{|\bar{s}_r|} \]  
(13)
In other words, the cross product of \( \psi \) and the projected state of stress gives the shear stress magnitude, and direction \( \psi \). It is insightful to note that
\[ \bar{S}_r = \tau_{\psi \psi} = \tau_{\psi \psi} \]  
(14)
Note, the overbar is used here to make it clear that the stresses are being evaluated at an arbitrary DCV.
Initial attempts to determine the normal and shear stresses of a complex stress tensor were not successful, but it was found that if one separates the complex stress tensor into its real and imaginary parts
\[ S = \text{Re}(S) + i \text{Im}(S) = S^R + iS^I \]  
(15)
then for an arbitrary DCV \( \psi \)
\[ \bar{s}_p^R = S^R \psi, \quad \bar{s}_p^I = S^I \psi \]  
(16)
\[ \bar{S}_N = \psi \bar{s}_p^R, \quad \bar{S}_N^I = \psi' \bar{s}_p^I \]  
(17)
\[ \bar{s}_r^R = \psi \times \bar{s}_p^R, \quad \bar{s}_r^I = \psi \times \bar{s}_p^I \]  
(18)
It is important to note that these real and imaginary stress components are all evaluated at a single \( \psi \). The real and imaginary shear stress directions \( \psi^R, \psi^I \) are
\[ \psi^R = \frac{\bar{s}_r^R}{|\bar{s}_r^R|}, \quad \psi^I = \frac{\bar{s}_r^I}{|\bar{s}_r^I|} \]  
(19)
One can also determine the magnitude of the complex shear and normal stresses by
\[ |\bar{S}_N| = \sqrt{\bar{S}_N^R \bar{S}_N^I}, |\bar{S}_r| = \sqrt{\bar{S}_r^R \bar{S}_r^I} \]  
(20)
given the magnitudes of shear stresses
\[ |\bar{s}_r^R| = \bar{S}_r^R, |\bar{s}_r^I| = \bar{S}_r^I \]  
(21)
The key point is that the magnitudes of the shear and normal stresses are computed based on real valued stress tensors \( S^R, S^I \) which maintain the sign relationships between their components at an arbitrary DCV \( \psi \). The use of magnitudes of the shear and normal stresses is satisfactory given that we are looking for the maximum value of these periodic components.

5.3 Stress tensor frequency response functions for multiple inputs
The general multi input random frequency domain response equation for stress is [14]
\[ G_{SS}(f) = \sum_{a=1}^{n} \sum_{b=1}^{n} H_{as}(f) H_{bs}(f) G_{ab}(f) \]  
(22)
where \( G_{SS}(f) \) is the PSD of stress response, \( H_{as}(f) \) is the complex frequency response function (FRF) for an output stress given an input \( a \), and \( G_{ab}(f) \) is the cross PSD (CPSD) of inputs \( a \) and \( b \). When \( b = a \), \( G_{aa}(f) \) is the PSD of input \( a \). For this discussion of multi axial loading particular attention needs to be given to the CPSD; there are several general cases
- CPSD = 0; The inputs are mutually uncorrelated
- CPSD > 0, real; The inputs are correlated and in phase
- CPSD < 0, real; The inputs are correlated and 180° out of phase
- CPSD > 0, imaginary; The inputs are correlated and 90° out of phase
- CPSD is complex; The inputs are correlated with an arbitrary phase relationship

Now, extend this to evaluate a complex stress tensor FRF $H_{as}(f)$ for a PSD input $a$.

Following the discussion above about complex stress tensors, define a complex stress tensor FRF for an input $a$ by its real and imaginary parts

$$H_{as} = \text{Re}(H_{as}) + i \text{Im}(H_{as}) = H_{as}^R + iH_{as}^I$$

where the function of frequency notation is omitted for clarity. It can be shown that eq (22) can be written as

$$G_{ss} = \sum_{a=1}^{n} \sum_{b=1}^{n} \left[ H_{as}^{R} H_{bs}^{R} + H_{as}^{I} H_{bs}^{I} \right] G_{ab}$$

(24)

A logical extension of this is to evaluate the stress tensor response function for a given DCV

$$\bar{G}_{ss} = \sum_{a=1}^{n} \sum_{b=1}^{n} \left[ \bar{H}_{as}^{R} \bar{H}_{bs}^{R} + \bar{H}_{as}^{I} \bar{H}_{bs}^{I} \right] G_{ab}$$

(25)

and a PSD of shear and normal stresses for this DCV as

$$\bar{G}_{\tau \tau} = \sum_{a=1}^{n} \sum_{b=1}^{n} \left[ \bar{H}_{as}^{R} \bar{H}_{bs}^{R} + \bar{H}_{as}^{I} \bar{H}_{bs}^{I} \right] G_{ab}, \quad \bar{G}_{\tau N} = \sum_{a=1}^{n} \sum_{b=1}^{n} \left[ \bar{H}_{as}^{R} \bar{H}_{bs}^{R} + \bar{H}_{as}^{I} \bar{H}_{bs}^{I} \right] G_{ab}$$

(26)

based on equations (16) thru (18)

5.4 **Determine the maximum shear and normal stresses and moments**

Following the method outlined by Pitoiset, et al [13], one can now determine the maximum shear and or normal stresses for a given element. Given that we are now working with a complex multi input problem, one must evaluate the shear and normal stress results for a range of DCVs that span the 3D sphere around a given point.

The key is that one needs to find the DCV for each element that gives the maximum shear and or normal stresses when one evaluates the output stress PSDs $\bar{G}_{\tau \tau}, \bar{G}_{\tau N}$ over the load frequency range. As presented in [13] it is also important to find the spectral moments of the PSDs for use in the failure models.

6 **CONCLUSION**

Wider use of FEA methods has made multi axial fatigue analysis more common in engineering design. Recently spectral techniques have been used, but these pose problems of non-proportionality. Two proven ways of dealing with this for deterministic load cases are strain path analysis and the concept of equivalence. The paper examines a limited number of published papers dealing with these. Strain path analysis seems too computationally intensive for spectral use. Developments in the manipulation of stress tensors has overcome many of the difficulties for spectral equivalence methods, and made them fast and efficient design tools. The paper then proposes new ways to evaluate the normal and shear stress principal planes which are of most relevance to the multi axial fatigue life calculation.

**REFERENCES**


