AN ANALYTICAL STUDY OF NON-LINEAR BEHAVIOUR OF COUPLED 2+2×0.5 DOF ELECTRO-MAGNETO-MECHANICAL SYSTEM BY THE METHOD OF MULTIPLE SCALES

Radoslav Darula and Sergey Sorokin

1Department of Mechanical and Manufacturing Engineering
Aalborg University
Fibigerstraede 16, 9220 Aalborg, Denmark
E-mail: dra@m-tech.aau.dk

Keywords: Coupled non-linear model, method of multiple scales, dynamics

ABSTRACT

An electro-magneto-mechanical system combines three physical domains - a mechanical structure, a magnetic field and an electric circuit. The interaction between these domains is analysed for a structure with two degrees of freedom (translational and rotational) and two electrical circuits. Each electrical circuit is described by a differential equation of the 1st order, which is considered to contribute to the coupled system by 0.5 DOF. The electrical and mechanical systems are coupled via a magnetic circuit, which is inherently non-linear, due to a non-linear nature of the electro-magnetic force. To study the non-linear behaviour of the coupled problem analytically, the classical multiple scale method is applied. The response at each mode in resonant as well as in sub-harmonic excitation conditions is analysed in the cases of internal resonance and internal parametric resonance.

1. INTRODUCTION

In the electro-magneto-mechanical systems, different physical domains are coupled, i.e. models of each domain are interconnected. In the paper a lumped-parameter approach is used to describe the dynamical behaviour of such a system shown schematically in Fig. 1, where the mechanical system is coupled to two electrical ones by a corresponding magnetic fields.

The mechanical system is assumed to allow motion in two directions (a vertical translation of the centre of gravity $w(t)$ and rotation around centre of gravity $\theta(t)$, Fig. 1). Such 2DOF mechanical system can be described by two differential equations of the second order.

The electrical system is assumed to be of LR type (inductance-resistance, as shown in Fig. 1), i.e. a capacitance of the system is neglected. Then each electrical system is described by a differential equation of the first order, i.e. it is of a diffusion type, contributing just with 0.5 DOF to the coupling system [1].

The electrical and mechanical systems are coupled by a magnetic system, which is composed only of reluctance and source terms (Fig. 1). The coupling is performed by an electro-magnetic force in the mechanical system and by an inductance term in the electrical system.
Figure 1. Schematics of the coupled 2+2×0.5DOF electro-magneto-mechanical system

In general, both coupling elements, i.e. the electromagnetic force and the electrical inductance are of non-linear nature. In order to solve the problem analytically, a perturbation analysis by means of the method of multiple scales [2] is applied.

2. MATHEMATICAL MODEL OF THE ELECTRO-MAGNETO-MECHANICAL SYSTEM

The analysed system is composed of (Fig. 1):

- 2DOF mechanical model, which represents a structure of mass $m_{m}$ and moment of inertia $J_{0}$ suspended on two spring-damper systems (characterized by a spring stiffness $k_{S,n}$ and a damping constant $c_{S,n}$) with electromagnetic elements ($k_{M,n}$). The position of centre of gravity (point C.G. in Fig. 1) is described by lengths $l_{1}, l_{2}$.

- Two magnetic circuits with air gap ($R_{a,n}$) and core ($R_{c,n}$) reluctances. The source of magnetic flux is a magnetomotive force from an electromagnetic coil $N_{w,n}i_{n}$.

- Two electrical RL-circuits, which represent two electromagnets. The electrical resistance $R_{S,n}$ serves as a dissipative component and the coupling is done via an inductance component $L_{m} = L_{s,n}(i_{AC,n}) + L_{m,n}(\dot{x}_{n})$, which includes a self inductance ($L_{s,n}$) and a speed-voltage component ($L_{m,n}$) [3]. The electrical circuit is supplied with the DC voltage $u_{DC}$.

To describe the dynamic behaviour of the coupled system, following coordinates are used $x_{S} = x_{S}(t)$, $\theta_{S} = \theta_{S}(t)$ and $i_{AC,n} = i_{AC,n}(t)$ (with $n = 1..2$). The system is supposed to be excited by a time-harmonic force $F = F(t)$ and/or a moment $M = M(t)$.

The electromagnetic force is derived using a lumped parameter approach [4] as

$$F_{n,m} = \frac{C_{e}(i_{DC,n} + i_{AC,n})^{2}}{(C_{0,n} + x_{S} + l_{n}\theta_{S} - h_{st,n})^{2}}$$

where $C_{e}$ and $C_{0,n}$ are constants of an electromagnet used and $i_{DC,n}$ is DC supply current, electromagnet is excited with. In the static case (i.e. for $x_{S} = 0$, $\theta_{S} = 0$ and $i_{AC,n} = 0$), the DC current induces a static deflection $h_{st,n}$.

2.1 Coupled equations

Let us first analyse the mechanical system. The dynamical response of a 2DOF mass-spring-damper system with the electromagnetic forces considered to act at the same points as springs/dampers (Fig. 1) is described as follows

$$m_{M}\ddot{x}_{S} + \sum_{n=1}^{2}\left(c_{S,n}(\dot{x}_{S} + l_{n}\dot{\theta}_{S}) + k_{S,n}(x_{S} + l_{n}\theta_{S} - h_{st,n}) + \frac{C_{c}(i_{DC,n} + i_{AC,n})^{2}}{(C_{0,n} + x_{S} + l_{n}\theta_{S})}\right) = F$$

841
\[ J_0 \ddot{\theta}_S - \sum_{n=1}^{2} \left[ (-1)^n \left( c_{S,n} l_n(x_S + l_n \dot{\theta}_S) + k_{S,n} l_n(x_S + l_n \theta_S) - h_{st,n} + \frac{C_{h,n} (i_{DC,n} + i_{AC,n})}{(C_{h,n} + l_n \dot{\theta}_S)^2} \right) \right] = -M \tag{3} \]

where the overdot indicates a derivative with respect to time \( t \) and \( C_{h,n} = C_{0,n} - h_{st,n}. \)

Each electrical system consists of a RL circuit, i.e. a resistance and an inductance term, which leads to

\[ u_{DC,n} = R_{S,n} (i_{DC,n} + i_{AC,n}) + \left( L_{i,n} + \frac{2 C_e}{(C_{h,n} + x_n)} \right) i_{AC,n} - \frac{C_e (i_{DC,n} + i_{AC,n})}{(C_{h,n} + x_n)^2} \frac{d}{dt} x_n \tag{4} \]

for \( n = 1..2. \)

Equations (2-4) constitute the system of equations describing the dynamic behaviour of the coupled electro-magneto-mechanical device.

### 3. ANALYSIS OF A COUPLED NONLINEAR PROBLEM

The coupled non-linear problem is defined by the system of two mechanical (Eqs. (2-3)) and two electrical equations (Eqs. (4) for \( n = 1..2 \)). As noted, the electromagnetic force (in mechanical equations) and inductance terms (in electrical one) are of non-linear nature.

A solution of the coupled problem is searched analytically by means of the method of multiple scales. This perturbation method is suitable for analysis of responses of weakly non-linear systems oscillating in the vicinity of an equilibrium point [2].

Since the multiple scale method is in general applicable for differential equations in the polynomial form, the non-linear terms are expanded in the vicinity of an equilibrium point to Taylor series. The problem is then solved assuming a weakly non-linear behaviour of a system, i.e. it is split into a linear (dominant) part and a non-linear one, responsible for modulation of amplitudes due to presence of non-linearities.

#### 3.1 Equilibrium point search

The equilibrium point is searched analysing the statics. Neglecting all dynamical terms in Eqs. (2-4), the coupled system becomes

\[ \sum_{n=1}^{2} \left( k_{S,n} h_{st,n} - \frac{C_r i_{DC,n}}{(l_{0,n} + l_0/2 - h_{st,n})^2} \right) - m g = 0 \tag{5} \]

\[ - \sum_{n=1}^{2} \left[ (-1)^n k_{S,n} l_n h_{st,n} - (-1)^n l_n \frac{C_r i_{DC,n}}{(l_{0,n} + l_0/2 - h_{st,n})^2} \right] = 0 \tag{6} \]

\[ u_{DC,n} = R_{S,n} i_{DC,n} \tag{7} \]

Eq. (7), an electric equation, provides the relationship between the supply voltage (in the case of voltage source used) and DC supply current.

Eqs. (5-6) can be rearranged so that a system of two polynomial equations of the third order in static deflections at each electromagnet (\( h_{st,n}, n = 1..2 \)) is obtained. Solving the system of the equations for \( h_{st,n} \), nine (in general complex valued) roots for static deflection at each electromagnet are found, as shown in Fig. 2a. Since \( h_{st,n} \leq h_{0,n} \) (which means that the yoke of magnet can move freely within the air gap), just roots 1-3 has a physical meaning. Roots 7-9 are unstable and roots 4-6 physically not feasible (since they fall in region \( h_{st,n}/h_{0,n} > 1 \), which would mean moving inside material) [5].

Taking the stable and physically feasible root, to simplify the solution, let us assume that \( h_{st,n} << (h_{0,n} + l_0/2) \) (which in practical applications is a case). Then using Taylor’s expansion, the static deflection found in the form

\[ h_{st,n} = \frac{C_n (C_r i_{DC,n} (\sum_{q=1}^{2} l_q) + l_{3-n} C_n^2 m g)}{(C_n^2 k_{S,n} - 2 C_r i_{DC,n} (\sum_{q=1}^{2} l_q))} \tag{8} \]

with \( C_n = h_{0,n} + 1/2l_0. \)
In order to analyse a set of non-linear design parameters of a core and a coil, two parameters can be used - an initial air gap \( h_{0,n} \) and a DC supply current \( i_{DC,n} \).

### 3.2 Limits of operation conditions

In general to control the static performance of an electro-magnetic element (apart from the design parameters of a core and a coil), two parameters can be used - an initial air gap \( h_{0,n} \) (i.e. a magnetic reluctance term) or a DC supply current (i.e. an magnetomotive force).

In Fig. 2a, there is a point designated as \( (i_{DC,n,1}, h_{0,n,1}) \), where roots 1-3 and roots 7-9 become complex valued, i.e. it corresponds to the limit point. From the analytical expressions of these roots, limit values of the operational parameters, i.e. current \( (i_{DC,n}) \) and initial air gap \( (h_{0,n,1}) \) are

\[
i_{DC,n} = \frac{1}{\sqrt{54C Sachs\, k_{s,n}}} \left( \frac{k_{s,n}}{2h_{0,n} + l_0 + l_1 + l_2 - 2h_{0,n}m_{an}} \right)^{3/2} \tag{9}\n\]

\[
h_{0,n,1} = \frac{h_{0,n}}{2} + \frac{3C_{n,2}^2 k_{s,n}^4 (l_1 + l_2)^6}{(2C_{n,2}k_{s,n}^2 S_{n,2}(l_1 + l_2)^2)^n} + \frac{i_{s,n,3}m_{an}}{k_{s,n}^2} \tag{10}\n\]

Fixing the value of air gap \( (h_{0,n}) \), the DC current can increase until the maximum allowable value, defined by Eq. (9) is reached. The dependence of the maximum current \( i_{DC,n,1} \) on initial air gap \( h_{0,n} \) is shown in Fig. 2b. Increasing the air gap, larger value of current can be applied to the electromagnets before the maximum allowable value is reached.

On the other hand, in the case of fixed current value \( i_{DC,n} \), the initial air gap can be reduced up to the minimum allowable value governed by Eq. (10). As presented in Fig. 2c, increasing the current, larger initial air gap values need be used before reaching a limit value. These outcomes are useful in practical applications of the coupled system.

### 3.3 Expansion of non-linear terms

In order to analyse a set of non-linear differential equations by the multiple scales method, the non-linear terms are to be expressed in a polynomial form. It means that electromagnetic force and inductances are expanded by Taylor’s expansion in the vicinity of an equilibrium point \( (h_{st,1,n}) \)

\[
f \left( \frac{C_{n,2}^2 i_{DC,n} + i_{AC,n}^2}{C_{n,2}^2 + i_{AC,n}^2} \right) \approx C_{n,2} \frac{C_{n,2}^2 i_{DC,n} + i_{AC,n}^2}{C_{n,2}^2} \left( 1 - \frac{2(x_S + l_0)k_{k,n}}{C_{n,2}^2} + \frac{3(x_S + l_0)k_{k,n}}{C_{n,2}^2} - \frac{4(x_S + l_0)k_{k,n}^3}{C_{n,2}^2} \right) \tag{11}\n\]

\[
\approx \sum_{n=1}^{2} \frac{2C_{n,2} i_{AC,n}^2}{C_{n,2}^2} \approx \frac{2C_{n,2} i_{AC,n}^2}{C_{n,2}^2} \left( 1 - \frac{(x_S + l_0)k_{k,n}}{C_{n,2}} \right) \approx \frac{2C_{n,2} i_{AC,n}^2}{C_{n,2}^2} \left( 1 - \frac{x_S}{C_{n,2}} - \frac{l_0k_{k,n}}{C_{n,2}} \right) \tag{12}\n\]

\[
\approx \frac{2C_{n,2} i_{AC,n}^2}{C_{n,2}^2} \left( 1 - \frac{x_S}{C_{n,2}} - \frac{l_0k_{k,n}}{C_{n,2}} \right) \approx \frac{2C_{n,2} i_{AC,n}^2}{C_{n,2}^2} \left( 1 - \frac{x_S}{C_{n,2}} - \frac{l_0k_{k,n}}{C_{n,2}} \right) \tag{13}\n\]
where the products of dynamic terms (i.e. terms $x_S \theta_S, x_S i_{AC,n}, \theta_S i_{AC,n}$ and their powers) are neglected, i.e. just the first three powers of each of dynamic terms are considered.

### 3.4 Multiple scales method

According to the method of multiple scales [6, 7], the dynamic equations are split into linear (leading) and non-linear (modulation) parts. The leading one is governed by a time scale $T_0 = t$ and modulated by a second time scale $T_1 = \varepsilon t$, with a book-keeper $\varepsilon$ used to mark small terms.

Then the small terms in Eqs. (2-4), with non-linear terms expanded according to Eqs. (11-13), written in the matrix-vector form are also book-kept by $\varepsilon$

\[
\begin{align*}
M\dot{x} + Kx + I_1 i + \varepsilon \left( I_2 \dot{\varepsilon}^2 + Cx + \sum_{n=1}^{3} C_n x^n \right) &= \varepsilon f \\
Ri + T(1 + \varepsilon X)\dot{x} + L(1 + \varepsilon X)i &= 0
\end{align*}
\]

where $M, K, C$ are mass, spring, damper constant matrices, $I_n, C_n$ are matrices of electromagnetic constants, $R$ is a matrix of resistances, and $T, L$ are matrices of inductances. Elements of all these matrices are defined by Eqs. (2-4, 11-13). For brevity, we do not display them here. A vector of harmonic forces is $f = [F, M]^T$ and $X$ is a $2 \times 2$ matrix of the form $X = [x_S, x_S; \theta_S, \theta_S]$. The vectors $x = [x_S, \theta_S]^T, i = [i_{AC,1}, i_{AC,2}]^T$ are vectors of displacement and current coordinates.

It is worth to notice, that due to a dependence of the natural frequency on the electric parameters, analysed e.g. in [8], the real part of the linear current term is not bookmarked by $\varepsilon$.

According to the method, the solution is expressed by an expansion of the coordinates

\[
\begin{align*}
x &= x_0(T_0, T_1) + \varepsilon x_1(T_0, T_1) \\
i &= i_0(T_0, T_1) + \varepsilon i_1(T_0, T_1)
\end{align*}
\]

with $x_0 = x_0(T_0, T_1), i_0 = i_0(T_0, T_1)$ being vectors of leading order linear response of displacement and current, respectively and $x_1 = x_1(T_0, T_1), i_1 = i_1(T_0, T_1)$ vectors of the first-order correction of displacement and current emerging from non-linear terms.

### 3.5 Linear terms ($O(\varepsilon^0)$)

Collecting the leading order terms, the system of two coupled linear ODEs is obtained

\[
\begin{align*}
M\dot{x}_0 + Kx_0 + I_1 i_0 &= 0 \\
Ri_0 + T\dot{x}_0 + L\dot{i}_0 &= 0
\end{align*}
\]

The first matrix equation (Eq. (18), a mechanical one) is of the second order, i.e. it governs harmonic free steady-state oscillations of the coupled system. The second matrix equation (Eq. (19), an electrical one) is of the first order, i.e. a diffuse type, which can contribute to dynamic response of the system only with an purely imaginary eigenvalue.

Let us assume a solution in the form [9]

\[
\begin{align*}
x_0(T_0, T_1) &= X_0 e^{i\omega t} + \bar{X}_0 e^{-i\omega t} \\
i_0(T_0, T_1) &= I_{AC0} e^{j\omega t} + \bar{I}_{AC0} e^{-j\omega t}
\end{align*}
\]

where $X_0 = [X_0, \Theta_0]^T$ and $I_{AC0} = [I_{AC,1}, I_{AC,2}]^T$ with complex conjugates marked by an overbar.
Substituting Eqs. (20-21) into an electrical equation (Eq. (19), the coupling of electrical and mechanical domains is established in the form

\[
\mathbf{I}_{AC0} = (j\mathbf{R} - \omega\mathbf{L})^{-1} (\omega\mathbf{T})\mathbf{X}_0 = \mathbf{I}_{AC0re}\mathbf{X}_0 + j\epsilon\mathbf{I}_{AC0im}\mathbf{X}_0
\]  
(22)

i.e. the electrical circuit contribute to vibration response of the mechanical system as an added stiffness (i.e. the real part \(\mathbf{I}_{AC0re}\mathbf{X}_0\) is linearly proportional to displacement \(\mathbf{X}_0\)) and damping (the imaginary part \(j\epsilon\mathbf{I}_{AC0im}\mathbf{X}_0\)). The influence of damping is considered to be weak, i.e. it is marked by \(\epsilon\).

Since the denominators of elements of the matrix \(\mathbf{I}_{AC0}\) defined by Eq. (22) are complex valued, the matrix is split into real and imaginary parts. This allows to distinguish contribution of linear part of electrical circuit to stiffness (real part of the matrix) and damping (imaginary part of the matrix). After separation of real and imaginary parts, we obtain

\(\mathbf{I}_{AC0re} = f(\omega^2), \mathbf{I}_{AC0im} = f(\omega^2),\) whereas \(\mathbf{I}_{AC0} = f(\omega).\) This modification adds two more (artificial) eigenvalues, which are complex conjugates to those of the each electrical system.

In order to find the eigenfrequencies of the linear system, let us substitute the real part of coupling current \(\mathbf{I}_{AC0re}\mathbf{X}_0\) to the mechanical equation (18). Then for existence of a non-trivial solution, the determinant should be equated to zero

\[
\begin{aligned}
\left(-\mathbf{M}\omega^2 + \mathbf{K} + \mathbf{I}_{AC0re}\right)\mathbf{X}_0 &= 0 \\
\det(s) &= \left|\mathbf{M}\omega^2 + \mathbf{K} + \mathbf{I}_{AC0re}\right| = 0
\end{aligned}
\]  
(23-24)

The order of determinant in \(\omega^2\) is four, i.e. four roots for \(\omega^2\) are obtained. The first two double roots for \(\omega\) correspond to natural frequencies of the coupled 2DOF mechanical system.

### 3.6 Natural frequencies

In general, two positive roots for \(\omega^2\) are found. Dependent on the location of the centre of gravity, described by the ratio \(l_2/l_1\), if \(l_2/l_1 = 1\) (i.e. perfectly centred mass) the first eigenfrequency corresponds to a pure translational motion (described by displacement of a centre of gravity \(x_S\)) and the second eigenfrequency to a pure rotational motion (described by rotation around the centre of gravity \(\theta_S\)), i.e. the motion of the mechanical system is described by a set of principal coordinates \((x_S, \theta_S)\). If the centre of gravity is misaligned, i.e. \(l_2/l_1 \neq 1\), the eigenfrequencies correspond to eigenmodes expressed as linear combinations of translational and rotational motion. Then coordinates are not principle ones, i.e. the motion of each mode is described by a translational component and a rotational one. The remaining purely imaginary eigenvalues are originated from the diffusion (electrical) equation.

As presented in Fig. 3, the natural frequency depends on two main dynamic operational parameters, namely an equivalent resistance \(R_{S,n}\) and a supply current \((i_{DC,n})\). The limiting cases are listed in Tab. 1, where \(\Omega_{n,k} = \omega_{n,k}/\omega_{n,1k}\) with \(k = 1..2, \omega_{n,k} a k-th\) eigenfrequency found from determinant in Eq. (24) and \(\omega_{n,1k}\) a natural frequency of linear system with no action of electromagnetic force (i.e. a conventional 2DOF mass-spring-damper system).

<table>
<thead>
<tr>
<th>Condition</th>
<th>Explanation</th>
</tr>
</thead>
<tbody>
<tr>
<td>(i_{DC} \rightarrow 0) and/or (R_S \rightarrow 0)</td>
<td>no influence of el. circuit, just response of linear 2DOF system, i.e. (\Omega_{n,k} \rightarrow 1, k = 1..2)</td>
</tr>
<tr>
<td>(i_{DC} \rightarrow \infty) and (R_S \rightarrow 0)</td>
<td>maximal detuning (maximal reduction of natural frequency)</td>
</tr>
<tr>
<td>Increasing (i_{DC})</td>
<td>Increasing detuning</td>
</tr>
<tr>
<td>Increasing (R_S)</td>
<td>Increasing detuning</td>
</tr>
</tbody>
</table>

Table 1. Limiting cases of operational parameters for eigenfrequencies.
Therefore, in the further study the condition $l_1/l_2 = 1$ can be seen that the influence of $o_{ff}$ (mode (Fig. 3a), the limit current of perfectly aligned system ($l_2/l_1 = 1$) detunes the system less than the system with off-set ($l_2/l_1 = 5$), i.e. the change of non-dimensional frequency ($\Omega_{n,1}$) is smaller as $R_{S,n} \to \infty$. On the other hand, for the second mode (Fig. 3a), smaller detuning is obtained for a system with off-set ($l_2/l_1 = 5$). However, the differences between maximal detuning at limit current values obtained for $l_2/l_1 = 1$ and $l_2/l_1 = 5$ are small. Therefore, in the further study the condition $l_2/l_1 = 1$ is assumed.

For practical purposes, the maximal detuning (i.e. region $i_{DC} \to i_{DC,t}$, $R_S \to \infty$) can be of interest. From the analysis a shift of eigenfrequencies by more than 30% for the first and 25% for the second eigenfrequency can be obtained.

### 3.7 Modulation terms ($O(\varepsilon^1)$)

The modulation terms, i.e. terms in Eq. (14) for the mechanical system bookmarked by order $O(\varepsilon^1)$, should be obtained from inhomogeneous equation

$$
\mathbf{M}\ddot{\mathbf{x}}_1 + \mathbf{K}\mathbf{x}_1 + \mathbf{I}_1i_1 = \mathbf{f} - 2\mathbf{M}\dot{\mathbf{x}}_0 - \mathbf{C}\mathbf{x}_0 - \mathbf{I}_1\mathbf{I}_{AC0m}\mathbf{x}_0 - \sum_{n=1}^{2} \mathbf{I}_n\mathbf{i}_n^m - \sum_{n=1}^{3} \mathbf{C}_n\mathbf{x}_0^n
$$

(25)

The zero order terms ($\mathbf{x}_0, \mathbf{i}_0$) are defined by Eqs. (20-21) and the relationship between the current and displacement is defined by Eq. (22). On the right hand side of Eq. (25) the following terms are recognized

- primary and higher order resonance terms - $e^{\pm jm\omega_0 T_0}$ with harmonic number $m = 1..5$ and mode $i = 1..2$;
- internal resonance terms - $e^{\pm j(2\omega_{n,1+2\omega_{n,2}})T_0}$, $e^{\pm j(3\omega_{n,1+2\omega_{n,2}})T_0}$, $e^{\pm j(4\omega_{n,1+2\omega_{n,2}})T_0}$ and reversed pattern.

It means that under certain relationships between natural frequencies $\omega_{n,1}, \omega_{n,2}$, the sub- or eventually super-harmonics can be excited, i.e. a non-linear modal coupling is established.

Furthermore, due to a connection between the mechanical and electrical systems, on top of modal coupling introduced by an internal resonance, an additional ‘electrical’ coupling is established as well. In the further analysis, first the fully de-coupled system is analysed (i.e.
governed purely by primary external resonance), next the modal coupling is added (while the system is electrically de-coupled) and finally a fully coupled system is considered.

The analytical results are compared with results obtained from a numerical model, generated by means of MATLAB Simulink. Due to some numerical issues, there is discrepancy between prediction for the indirectly excited mode and so the results for this mode are not presented in this paper. The parameters used in simulations are summarized in Tab. 2.

<table>
<thead>
<tr>
<th>Analysed system</th>
<th>$\omega_{n_1}/\omega_{n_2}$</th>
<th>$\zeta_1/\zeta_2$</th>
<th>$l_2/l_1$</th>
<th>$h_{0,1,2}$</th>
<th>$R_{S,1,2}$</th>
<th>$M_0$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Prim.res.full.decoupl.</td>
<td>201/403</td>
<td>0.4/0.06</td>
<td>0.5</td>
<td>3x10^5</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Subhar.elec.decoupled</td>
<td>403/806</td>
<td>0.4/1.1</td>
<td>1</td>
<td>0.5</td>
<td>1x10^-3</td>
<td>3x10^5</td>
</tr>
<tr>
<td>Subhar.mod.and el.cpl.</td>
<td>434/868</td>
<td>0.4/1.4</td>
<td>var</td>
<td>var</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Table 2. Parameters used in the study.

### 3.8 Primary resonance of fully de-coupled system

Considering just the primary resonance terms (i.e. $e^{\pm j\omega_0 t}$ terms) and exciting the system only by a force (in a case of the first eigenfrequency $\omega_{n,1}$) or a moment (in the second one $\omega_{n,2}$), the external primary resonant response can be determined.

According to the multiple scale method, the secular terms (i.e. terms containing $e^{\pm j\omega_0 t}$) have to be removed from right hand side of Eq. (25). Then for the force excitation (i.e. excitation at $\omega_{n,1}$), the secular terms are equated to zero

$$\Phi_1 e^{\omega T_1} + C_{r,1,1-1c}X_0^2 \dot{X}_0 + j \zeta_1 \omega_{n,1} \dot{X}_0 + j 2 C_{m,1,1} \omega_{n,1} \dot{X}_0 = 0$$

$$C_{r,1,2-2c} \Theta_0^2 \dot{\Theta}_0 + j \zeta_2 \omega_{n,2} \dot{\Theta}_0 + j 2 C_{m,2,1} \omega_{n,2} \dot{\Theta}_0 = 0$$

where $\Phi_1 = f(F)$ is a forcing term, $C_{r,i,j}, i = 1..2, j = 1..2$ are coefficients of 'electric' terms and $C_{m,i,j}, i = 1..2, j = 1..2$ are coefficients of 'mechanical' terms and $\zeta_0$ is damping coefficient (see Eqs. (2-4,11-15)).

It can be noticed that the first equation is excited by external forcing ($\Phi_1 e^{\omega T_1}$), whereas the second one is fully de-coupled with no excitation. In general the second equation leads into two possible solutions. Besides the trivial solution (i.e. $\Theta_0 = 0$), Eq. (27) may have a non-trivial solution ($\Theta_0 \neq 0$). Since the non-trivial solution is not externally excited (i.e. it is a case of free-vibration) and all initial conditions are treated to be zero, in further analysis the trivial solution is considered and Eq. (26) solved for the directly excited mode $X_0$.

Expressing the displacement in terms of real valued amplitude and phase functions of slow time [2, 6]

$$X_0(T_1) = A_{0,1}(T_1) e^{i\psi_1(T_1)}$$

after separation of real and imaginary parts, the modulation equation becomes

$$\Phi_1 \sin \theta_1 + \zeta_1 \omega_{n,1} A_{0,1} = 0$$

$$\Phi_1 \cos \theta_1 + \sigma_2 C_{m,1,1} \omega_{n,1} A_{0,1} + C_{r,1,1-1c} A_{0,1}^3 = 0$$

with $\theta_1 = T_1 \sigma_2 - \psi_1$.

Solution of Eqs. (29-30) for amplitude $A_{0,1}$ leads to a bi-cubic polynomial equation with three double, in general complex, solutions for $\omega_{n,1}$. That means that there can be found two stable and one unstable roots, as valid for conventional cubic non-linear systems.

The same procedure as for a force excitation at $\omega_{n,1}$ can be applied for the second mode (i.e. moment excitation at $\omega_{n,2}$). The response of the de-coupled system at each primary resonance
is presented in Fig. 4. Due to a presence of non-linear terms, the frequency responses at resonances undergo:

- softening at the first mode (excitation by force at $\omega_{n,1}$);
- softening at the second mode (excitation by moment at $\omega_{n,2}$).

In order to validate the analytical model, numerical simulations (MATLAB Simulink model) were performed. The comparison of analytical and numerical results are presented in Fig. 4, where agreement between the two method can be seen.

![Figure 4: Response of the system at resonant excitations of (a) the first mode ($\omega_{n,1}$), (b) the second mode ($\omega_{n,2}$). The analytical model (MMS) is compared with numerical simulations (Num.Sim.)](image)

3.9 Sub-harmonic excitation of electrically de-coupled system with internal parametric resonance

Next let us de-couple the system from electrical circuits, i.e. take $i_{AC} \rightarrow 0$ and consider a modal coupling in the form of internal parametric resonance (i.e. $\omega_{n,2} = 2\omega_{n,1} + \varepsilon\sigma_1$). If the system’s excitation is applied at $\Omega = \omega_{n,2} + \varepsilon\sigma_2$ (i.e. at the second mode), the first mode is excited parametrically.

The modulation equations of such system are

$$\Phi_2 \sin \theta_1 + \zeta_2 \omega_2 A_{0,2} = 0$$
$$\Phi_2 \cos \theta_1 + 2\sigma_2 C_{1,2,1}\omega_{n,2} A_{0,2} + C_{1,2,3}A_{0,2}^3 = 0$$
$$C_{2,1,2}A_{0,2}^2 \sin \theta_2 + \zeta_1 \omega_1 A_{0,1} = 0$$
$$C_{2,2,2}A_{0,2}^2 \cos \theta_2 + (\sigma_1 - 2\sigma_2)C_{m,1,1}\omega_{n,1} A_{0,1} + C_{2,1,3}A_{0,1}^3 = 0$$

with forcing term $\Phi_2 = f(M)$ and coefficients of polynomial terms $C_{i,j,k}, i = 1..2, j = 1..2, k = 1..2$.

After some manipulations, the system of equations (31-34) can be re-formulated into the set of polynomial equations

$$C_{2,4}A_{0,2}^4 + C_{1,2}A_{0,1}^2 + C_{1,4}A_{0,1}^4 + C_{1,6}A_{0,1}^6 = 0$$
$$\Phi_2^2 + C_{2,2}A_{0,2}^2 + C_{2,4}A_{0,2}^4 + C_{2,6}A_{0,2}^6 = 0$$

Equation (36), containing also forcing terms, is fully de-coupled and governs motion of directly excited mode ($A_{0,2}$). The indirectly excited mode ($A_{0,1}$) is coupled with $A_{0,2}$ via Eq.(35). Considering $A_{0,1} \rightarrow 0$, the response of fully de-coupled system is recovered.
Figure 5: (a) Frequency responses of electrically de-coupled system with internal parametric resonance and sub-harmonic excitation: an analytical model compared with numerical simulations; (b) Comparison of electrically de-coupled and coupled model with large resistance ($R_{s,n} = 500\,\text{k}\Omega$)

Figure 6: Frequency response functions for fully coupled model and (a) the resistance variation (and $i_{DC,n} = 0.5\,\text{A}$); (b) the electric current variation (and $R_S = 50\,\text{k}\Omega$)

Solving the set of equations (35-36), amplitudes can be found. As presented in Fig. 5, the directly excited mode (the second mode at $\omega_{n,2}$) retains its softening behaviour, similarly as for fully de-coupled system.

### 3.10 Sub-harmonic excitation with modal and electrical coupling

In the previous cases the electrical circuit was disconnected and the modal interaction for electrically de-coupled system was observed. To add another coupling, the electrical circuit is assumed to be connected (i.e. $i_{AC,n} \neq 0$) and influence of both modal as well as electrical coupling is analysed.

Adding also terms containing electric current, the modulation equations are modified to

$$
\Phi_2 \sin \theta_1 + \zeta_2 \omega_2 A_{0,2} + C_{1,2,1} A_{0,2} = 0
$$

$$
\Phi_2 \cos \theta_1 + \sigma_2 C_{1,2,2} \omega_{n,2} A_{0,2} + C_{1,2,1} \omega_{n,2} A_{0,2} + C_{1,2,3} A_{0,2}^3 = 0
$$

$$
C_{2,1,2} A_{0,2}^2 \sin \theta_1 + \zeta_1 \omega_1 A_{0,1} + C_{2,1,1} A_{0,1} = 0
$$

$$
C_{2,2,2} A_{0,2}^2 \cos \theta_2 + (\sigma_1 - 2\sigma_2) C_{m,1,1} \omega_{n,1} A_{0,1} + C_{2,0,1} A_{0,1} + C_{2,1,3} A_{0,1}^3 = 0
$$

where the forcing term $\Phi_2 = f(M)$ and $C_{i,j,k}, i = 1..2, j = 1..2, k = 1..2$ are coefficients of polynomial terms.
Amplitudes of vibration are found re-formulating Eqs. (37-40) into a set of polynomial equations, similar to Eqs. (35-36), with coefficients $C_{jk}, j = 1..2, k = 1..2$, governed by Eqs. (37-40).

The frequency responses are presented in Fig. 6. As introduced in Sec. 3.6, there are two operation parameters in the electrical circuit - DC current $i_{DC,n}$ and an equivalent resistance $R_{S,n}$.

Fixing the DC current ($i_{DC,n} = 0.5A$), the variation of resistance $R_{S,n}$ influences the added damping, which is introduced by a coupling to electrical circuit. It can be seen, that there is a resistance value at which the amplitude at resonance is smallest and also the non-linear behaviour is fully suppressed (Fig. 6).

If the resistance value $R_{S,n} \to \infty$, then $i_{AC,n} \to 0$ and the electric circuit is not contributing to the dynamical response of the system, i.e. the system is electrically de-coupled, as discussed in Sec. 3.9. A comparison of the two cases (electrically de-coupled and fully coupled with large resistance value, $R_{S,n} = 500k\Omega$) is shown in Fig. 5b.

On the other hand, keeping the resistance level constant (at value $R_S = 50k\Omega$) the variation of electric current, apart of detuning control discussed in Sec. 3.6, controls also the level of non-linearity, as presented in Fig. 6b.

4. CONCLUSIONS

In the paper dynamics of the coupled 2+2×0.5 DOF weakly non-linear electro-magneto-mechanical system was analysed. The problem was solved applying the method of multiple scales and vibrations in the vicinity of the equilibrium point were analysed both for decoupled and coupled (mechanically and electrically) systems.

From the linear solution the detuning of eigenfrequencies, due to coupling between the mechanical system and the electrical circuit, was observed. The level of detuning is a function of electrical parameters (namely a DC current and electrical resistance). Increasing these parameters, the natural frequency is detuned to lower (for the first) or higher frequencies (the second mode). As shown, significant detuning can be reached (more then 30% for the first mode).

Analysing the fully coupled system (i.e. considering internal parametric resonance as well as electrical coupling), the influence of electrical circuit was assessed. Introduction of additional (so-called electrical) damping was observed.

Both detuning and damping concepts can in practice be used in vibration control. As the amount of damping as well as detuning can be controlled by electrical parameters, the active or adaptive-passive control scheme can be used to reduce an excessive vibration by means of electro-magneto-mechanical systems.

ACKNOWLEDGMENTS

The article is supported by InterReg IV A - the Silent Spaces Project.

REFERENCES


