DYNAMIC INSTABILITY IN AN APPARENTLY SIMPLE EXPERIMENTAL STRUCTURE

Richard Wiebe\textsuperscript{1}, Lawrence N. Virgin\textsuperscript{1∗}, Stephen M. Spottswood\textsuperscript{2}, and Thomas G. Eason\textsuperscript{2}

\textsuperscript{1}School of Engineering
Duke University
Durham, NC 27708, U.S.A.
∗E-mail: l.virgin@duke.edu

\textsuperscript{2}Structural Sciences Center,
Aerospace Systems Directorate,
Air Force Research Lab, WPAFB, OH, U.S.A

Keywords: snap-through buckling, nonlinear vibrations, experimental mechanics

ABSTRACT

Slender structures with softening force-deflection characteristics are liable to exhibit snap-through buckling. This type of thoroughly nonlinear static behavior profoundly influences the situation in which the lateral loading is applied dynamically. In the case of harmonic loading the amplitude and frequency of the force are the key parameters influencing snap-through. Although this is primarily a transient effect, the phenomenon of resonance may appear. Much of the previous research in this area is related to single-degree-of-freedom systems. The current presentation focuses attention on this type of behavior in very thin shallow arches from an experimental perspective, although a limited set of results from finite element analysis are included for verification purposes. For this kind of continuous system there is a surprisingly wide variety of possible behavior, e.g., the snap-through may occur in a symmetric or asymmetric type of mode, various higher-order periodic response, a sensitive dependence on initial conditions, all of which occur about and around potentially complicated equilibrium paths and corresponding natural frequencies. The key results are associated with establishing appropriate definitions of snap-through as well as delineating regions within parameter space where snap-through dynamics are likely to occur. The practical context for this work relates to the anticipation of sonic fatigue in thin curved panels typically used as structural components in aircraft structures.

1. INTRODUCTION

The occurrence of snap-through of continuous curved structural systems is usually something to be avoided, especially in the fatigue-severe context of aircraft structures. Although there are certain useful applications such as the satisfying sudden clunk when pushing button, or the cap used to ensure the integrity (in a non-tampering sense) of a sealed food container. Even flat panels may undergo snap-through, for example if they are buckled by thermally induced
axial loads. The nonlinear dynamics of buckled and curved panels under acoustic loading is studied in [1, 2], including a limited amount of experimental study. The development of snap-through boundaries (as a function of forcing parameters) for shallow arches is the primary goal of this paper, with the focus squarely on experimental results.

Although curved panels occur widely in aircraft structures, we focus here on shallow arches. They retain the fundamental snap-through behavior of more complex and realistic structural components but are, of course, geometrically much simpler. The stability of shallow arches under various (load and boundary) conditions has been widely studied [3–13] and it has been shown that chaos can be an important aspect of the response. More Recent work in [14, 15] produced numerical lower bounds on snap-through with respect to thermal loading.

In this paper, the static response of shallow arches is studied under point loading and the dynamics under distributed harmonic excitations. A schematic of the arch and its properties is shown in Fig. 1. It is relatively well known that buckling behavior can be quite sensitive to small changes in the geometry or loading or boundary conditions of the system (as well as relatively small changes in the surrounding thermal environment). In fact, in the static results to be presented we also show the effect of moving the point load slightly away from the center (from $F_c$ to $F'_c$ in Fig. 1(a)) to illustrate the role of asymmetry. The two loading scenarios (static point and dynamic distributed) was partly due to experimental convenience, but this also serves to show that snap-through is a typical response regardless of the exact form or distribution of the lateral loading. The experimental results compare with limited FEA simulations very well. Details of the FEA can be found elsewhere [15]. Many more results related to the current study can be found in the dissertation [16].

2. EXPERIMENTAL SET-UP

A photograph of the structure under investigation is shown in Fig. 2. The (steel) arch was clamped to a support beam which was itself attached to the white circular drum of the shaker which provided the distributed dynamic load via the inertia of the arch itself. For the static testing the shaker was left off and used only as a mount for the arch. The static loading mechanism consisted of the point loading coupler attached to an adjustable threaded rod through a load cell. The threaded rod was moved perpendicular to the arch to displace the

![Figure 1: (a) Geometry of the shallow arch, with a typical digital image correlation (DIC) image shown above, (b) cross-section, and (c) mechanical properties.](image-url)
structure, while the load cell provided the corresponding point load magnitude. It is worth noting that only compressive forces were accessible as the loading was provided through contact. The static arch displacements were measured by DIC [17], which necessitated the speckle pattern visible in the photo.

![Figure 2: Photograph of shallow arch attached to shaker in clamped-clamped configuration; point loading mechanism used for static testing also visible.](image)

The dynamic tests were accomplished using the shaker in Fig. 2 with the static forcing apparatus removed. The shaker was capable of levels in excess of 14\( g \), and frequencies greater than 200 Hz, both of which were beyond what was needed to characterize the shallow shallow arch under study.

The majority of the response data was obtained with a laser vibrometer, however, dynamic DIC was used for several spot checks. The experiments were all performed at the U.S. Air Force Research Labs in Dayton, Ohio.

3. STATIC EQUILIBRIA AND STABILITY

The appearance of snap-through is a direct consequence of the nonlinear nature of the equilibrium (force-deflection) path. For certain levels of steady loading, multiple stable equilibria may exist with a solution path that contains intervals corresponding to unstable equilibrium configurations. These equilibria, both stable and unstable, play an important organizational role in the dynamic behavior.

Static force-displacement tests were performed respectively for a point loads \( F_C \) and \( F'_C \) at the midspan and offset 12.7 mm left of midspan respectively. The results, with the displacement measured (in fact the displacements at midspan were interpolated as will be discussed later) at midspan, are shown in Fig. 3. Given that the test was performed using displacement control it was possible to follow the downward component of the equilibrium paths, something that would not be possible with force control as these regions are unstable. However, the force was applied through a single sided ‘pushing’ contact, and could not carry tension, thus the arch did snap-through a short distance from point (iii) to (iv) once the equilibrium force crossed below the x-axis.

Interestingly, under midspan loading \( F_C \), the arch snaps-through in an asymmetric config-
Figure 3: Comparison of experimental (filled circles) and FEA (solid lines) static force-displacement relationship for displacement measured at midspan: (a) load at midspan $F_C$, (b) Load 12.7 mm left of midspan $F'_C$.

uration. This can be seen when referring to the full field displacements in Fig. 4 (a) for the four equilibrium states labeled (i) through (iv) in Fig. 3 (a). A practical issue arose in measuring the displacement at midspan due to the loading mechanism being placed coincident at this location which blocked the line of sight for the DIC cameras. The displacement in this region was therefore interpolated using an 8th order polynomial and the known data to each side of the missing data. However, the excellent agreement between the experimental and numerical displacements in the directly measured regions (filled circles) does provide some level of confidence that the interpolated displacements (empty circles) at the midspan are reasonable. The susceptibility of the structure to snap-through asymmetrically, i.e., favoring one side over the other, is dictated by initial imperfections in the structure. This preference can be heavily influenced by a modest shift in the load location as seen in Fig. 3(b), and Fig. 4(b). At some point a bifurcation occurs that causes the structure to snap-through more on the other side. The nature of the equilibrium path is, however, made much more complicated after this bifurcation diagram due to the imperfections.

Figure 4: Comparison of experimental (filled circles for measured, empty circles for interpolated) and FEA (solid lines) arch configurations for loads labeled (i) through (iv) in Fig. 3. (a) load at midspan $F_C$, (b) Load 12.7 mm left of midspan $F'_C$.

4. DYNAMIC SNAP-THROUGH

This section focuses on characterizing the dynamic behavior of the arches and dependence on the forcing parameters. Of relevance and providing significant insight into the transient behavior is the identification of the regions in the parameter space where the arch experiences snap-through under harmonic excitation, i.e., of the boundaries that separate the snap and
non-snap regions. The harmonic forcing in this case is inertial, i.e., the shallow arch and it’s support mechanism are subject to the motion of the shaker.

Figure 5: Experimental displacement (at midspan) time series and DFTs under harmonic forcing with $A_F = 1.8 \text{ N/m}$: (a) & (d) $f_F = 75 \text{ Hz}$ (noisy harmonic response), (b) & (e) $f_F = 130 \text{ Hz}$ (chaotic medium amplitude response), and (c) & (f) $f_F = 180 \text{ Hz}$ (chaotic snap-through response). In part (a) a typical forcing input is shown, the same units but displaced vertically for clarity.

Figure 5 (a), (b), and (c) show the DIC measured displacement of the arch under a harmonic distributed force of $1.8 \text{ N/m}$ at 75, 130, and 180 Hz respectively. The harmonic forcing is indicated by the superimposed curve in the upper part of Fig. 5(a) and shows that the response shown below in part (a), although slightly chaotic at most, does exhibit a primarily periodic nature. The biased nature of the response is expected, as the structure is very resistant to upward displacements, that is, the slope (stiffness) of the force-displacement relationship steepens sharply as the structure is raised upwards. Unfortunately, the dynamic DIC displacement measurements contained a small bias in the displacement readings (a single unknown shift from the datum at $t = 0$), hence the true datum ($d = 0$) may be shifted slightly from what is shown in parts (a) to (c). However, the peak-to-peak displacements provide enough information to determine snap-through. The normalized (by the power at the forcing frequency $F(f_F)$) DFTs of the time series in parts (a), (b), and (c) are shown in parts (d), (e),
and (f) respectively showing a transition from a rather narrow-band periodic to a wide-band chaotic response. Part (e) is especially interesting as the forcing frequency (180 Hz) is almost completely masked by the system.

The static equilibrium results obtained in the previous section provide a useful guide for the selection of the snap-through threshold. Figure 3 shows that, at midspan, the boundary separating the unsnapped from the snapped-through configurations is at approximately 4 mm when unloaded. Of course, with a changing load this threshold shifts harmonically with the applied forcing (inertial forces also add to the difficulty of relating static and dynamic snap-through), however, because the mean forcing is zero this is the most justifiable snap-through definition. The experimental time series shows that snap-through events are relatively easy to distinguish as they have significantly larger amplitude than non-snap oscillations. This facilitates an automated computation of the snap-through boundaries as well as highlighting snap-through as a somewhat extreme event (that designers would likely seek to avoid). Figure 6 (a) shows the experimental snap-through boundary in forcing parameter space ($f_F, A_F$), where circles indicate parameters for which the response exhibited snap-through. Each point is a single 30 s test. In order to minimize excess data, only the midspan response was measured using a laser vibrometer in lieu of full field measurements with the DIC system. The midspan was the most obvious choice for the test point and provides sufficient data to characterize snap-through, despite the difficulty in identifying other, more distributed, characteristics of the motion. The measured velocities were integrated and the snap-rate was determined from the peak-to-peak amplitudes (absolute displacement was lost due to a DC bias in the laser vibrometer). Part (b) is the equivalent numerical results (using FEA) over a slightly larger range of forcing parameters. Each point in the grid is a single simulation of 600 forcing cycles, with the ICs corresponding to the rest state. Other studies have indicated the important effect that initial conditions can have on snap-through, but starting from the rest state is certainly representative and natural for many real systems. Although the exact displacement was available numerically, the peak-to-peak values were similarly used to determine snap-through, thus allowing a consistent comparison with the experimental results.

Figure 6: Snap-through boundaries in forcing amplitude and frequency space showing chaotic snap-through (filled circles), chaotic non-snap (empty circles), and non-chaotic non-snap (small dots). (a) Experimental and (b) co-rotational FEA model. The gray x’s correspond to the parameter values time series in Fig. 5, the blue x’s to Fig. 7 and Fig. 8. The parameter range of part (a) is denoted by a dashed red square in part (b).
It is likely that comparing the peak-to-peak response with a snap-through definition based on the static equilibrium will yield a slightly conservative snap-through boundary. This is because a small component of the peak-to-peak response will be negative displacements away from snap-through. This is a consequence of the loss of the datum in the experimental testing, however, the system exhibits a very biased dynamic response with the majority of the displacement occurring in the direction of snap-through.

A second classification, that of chaotic non-snap responses is also shown in Fig. 6 as empty circles. The conventional classification of chaos is typically based on the computation of at least one positive Lyapunov exponent (LE) [18]. However, the determination of LEs for the experimental results proved difficult to apply due to the dimensionality of the system. Therefore, chaos was determined, for both the experimental and numerical time series, by use of the heuristic method developed in [19] which was shown to agree well with the more conventional LEs. It is based on counting the number of discrete peaks in the frequency spectrum, i.e., establishing the distinct difference in frequency content between a chaotic and non-chaotic signal. In this method the response is deemed chaotic if its DFT contains a sufficient number of peaks. That is, the number of peaks above a threshold (a fraction of the largest peak), say 5%, to avoid noise, are counted. If this number is relatively large, above a second threshold, then it is labeled chaotic. If the number of peaks in the DFT are relatively few, the response is labeled non-chaotic. However, the output is quite insensitive to the selection of this first threshold as the non-chaotic responses typically had less than 1% peaks while chaotic responses were closer to 15% or higher. It is worth noting that for a completely random sequence of data points, on average only 25% of them would be peaks, which makes this somewhat of an upper limit for the peak count.

The snap-through responses in this system were also all chaotic, something that did not occur in a related single-degree-of-freedom (SDOF) system in [20]. In that study it was found that non-chaotic large oscillation snap-through events were quite common, while chaotic non-snap events were rare. It is expected that the non-chaotic snap-through oscillations, although none were observed in tests, would almost surely be found for very large amplitude forcing where the arch would be obliged to snap-through in phase with the forcing. For the counteracting case of chaotic non-snap response, it is not surprising that a continuous arch, an infinite dimensional system, would be more likely to produce chaos for non-snap, i.e. single potential well (or hyper-well) oscillations, than a SDOF model.

Another interesting, and unexpected, phenomenon occurred in many of the chaotic snap-through responses in the upper right quadrant of Fig. 6(a) and (b), where the system would frequently get ‘stuck’ in the snapped-through configuration. A sample experimental velocity time series \( (A_F = 2.87 \text{ N/m}, f_F = 180 \text{ Hz}) \) in which ‘sticking’ occurred is shown (after about 2 seconds) in Fig. 7 as the region of low velocity response (the exact displacement is not known due to the DC bias). These parameters are in the snap-through region and are denoted by a blue x in Fig. 6(a) and (b). The system only exited this state when it was subjected to an external perturbation which was imparted by manually (lightly) striking the arch. After getting stuck the response was typically small amplitude periodic (period 1) as seen in the zoomed in inset in Fig. 7.

The sticking phenomena would have affected the chaotic vs. non-chaotic classification of a response. However, because the sticking response appeared to co-exist with a non sticking chaotic snap-through, it was decided to classify these responses based on their snap-through responses. This assumption was verified with the FEA model, as shown in Fig. 8. Parts (a) and (b) were both obtained using \( A_F = 2.87 \text{ N/m} \) and \( f_F = 180 \text{ Hz} \) (matching the experimental sticking sample), however, the initial forcing phases were set at different values. Similar to the experimental results, part (b) shows that after getting stuck in the snapped-through state, the system exhibits a small amplitude periodic (period 3 in this case) response. In order to
highlight the sensitive and indeterminate nature of the response, part (c) shows a series of 100 simulations with different initial forcing phases where $Y_s = 0$ indicates a response that persistently snaps-through and $Y_s = 1$ a response that gets captured in the snapped-through configuration, i.e., this latter phenomenon is less common.

The stability of small oscillations in the snapped-through configuration is especially surprising considering that the small amplitude non-snap response is highly unstable under the same forcing parameters. Under static loading the opposite is true, the non-snap configuration is more stable than the snapped-through state. This is demonstrated by the fact that the initial unloaded state is more distant from the separatrix at 4 mm than the snapped-through (but still unloaded) state in Fig. 3. There also exists the possibility of three co-existing re-
sponses near the snap-through boundary, that of persistent snap, small non-snap oscillations, and small oscillations about the snapped-through configuration.

5. CONCLUSIONS

In this paper the static and dynamic snap-through of a shallow arch is investigated, largely based on experimental means. The aim of the research is to develop a better understanding of the dynamics of nonlinear systems with the potential for snap-through. Static equilibria are determined first and later used to help distinguish between dynamic snap-through and non-snap-through. It is shown that dynamic snap-through is indeed distinguishable from non-snap response as it exhibits motion with much larger amplitudes than other non-snap responses. Beyond snap-through, the motion is also divided into regimes of chaotic and non-chaotic behavior. Chaos is shown to be inherently linked to snap-through as it occurs only within, or very near to regions of snap-through response.

The static equilibrium paths are found to be surprisingly complex, with many unstable equilibrium configurations co-existing between the snapped-through and non-snapped-through configurations. This undoubtedly contributes to the complexity of the dynamics of the system, since these unstable equilibria have a dominant effect in the global phase space and hence strongly influence transient and steady-state behavior.

ACKNOWLEDGMENTS

The work has been funded in part by AFOSR under the grant # FA9550-09-1-0201 and Universal Technology Center contract #12-S2603. The DIC data was acquired with the assistance of David Earhardt.

REFERENCES


