PGD-VTCR: A REDUCED ORDER MODEL TECHNIQUE TO SOLVE MID-FREQUENCY BROAD BAND PROBLEMS ON COMPLEX STRUCTURES

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ABSTRACT

The calculation of the vibrational responses of systems on frequency bands appears to be more and more important in engineering simulation. This is particularly true in medium frequency regimes, where these responses have a strong sensitivity to the frequency. To find this, standard techniques may involve many frequency fixed computations. In this work, we propose a novel technique that does not need the resolution of vibration problems at many different frequencies. It combines the use of the Variational Theory of Complex Rays (VTCR), a mid frequency dedicated numerical strategy, and the Proper Generalized Decomposition (PGD), a reduced order model technique. The VTCR uses exact solutions of the governing equation (propagative and evanescent waves) to find approximated solutions of the vibrational problem. The PGD expands the VTCR approximation over the frequency band thanks to a separated representation of the unknowns. Examples of such a strategy on complex structures will illustrate the efficiency of the proposed approach.

1. INTRODUCTION

On many industrial systems, numerical prediction techniques are being increasingly used because they limit the need for physical tests on prototypes to a minimum. In this context on aerospace examples and acoustic car design, engineers are often interested in the response of systems in frequency bands. The advanced calculation of the acoustic behavior of systems is the topic of this paper, in which we propose an approach to the development of frequency prediction techniques in frequency bands. It is based on a separated representation of the
unknown field, and uses the Variational Theory of Complex Rays (VTCR) [1] in combination with Proper Generalized Decomposition (PGD) [2].

The VTCR is a Trefftz method, and then uses exact solutions of the governing differential equations to expand the field variables. It is based on an original variational formulation of the problem which was developed in order to allow the approximations within elements to be \textit{a priori} independent of one another. Thus, in each element, any type of shape function can be used, as long as it verifies the governing equation. This property gives the approach great flexibility and, consequently, efficiency because every shape functions with a strong physical meaning related to the desired solution can be introduced without difficulty.

The decisive advantage of all Trefftz methods is the use of exact solutions of the governing equations. As a consequence, the model’s size and the computational effort is considerably reduced in comparison with element-based methods. The VTCR differs from other Trefftz methods (such as the partition of unity method [3], the ultra-weak variational method [4], the least-squares method [5], the discontinuous enrichment method [6] or the wave-based method [7]) by the way it handles the transmission conditions at the interelement boundaries and by the types of shape functions it uses. It was already successfully used on vibration problems involving 3D plate assemblies [8], shell structures [9] and acoustic problems [10].

The PGD is a model order reduction technique. It has already been successfully used for the resolution of multiparametric problems (problems which depend on many parameters such as the space and time problems, the space and uncertain problems, ...). This is the case for acoustic space problems for multiple frequencies. Therefore, a combination of PGD and the VTCR is an obvious choice to handle frequency problems in frequency bands.

The VTCR has already been adapted to frequency band applications in [11]). The authors proposed in these works new algorithms for the calculation of multiple-frequency solutions, either by using a set of parameters to derive a discrete approximation of the frequency-dependent quantities within the VTCR matrix, or by expanding the VTCR matrix and the right-hand side of the system to be solved into Taylor series with respect to the frequency. The objective of the technique we are proposing here is to develop new regards to frequency band analysis with the VTCR.

2. THE REFERENCE PROBLEM

Let us consider a 2D interior acoustic problem of a bounded domain $\Omega$ filled with a fluid characterized by its sound velocity $c_0$ and density $\rho_0$, and the frequency interval $I = \left[ \omega_0 - \frac{\Delta \omega}{2} ; \omega_0 + \frac{\Delta \omega}{2} \right]$ ($\omega_0$ denotes the central frequency and $\Delta \omega$ the bandwidth of the frequency band being considered). The reference problem to solve is: find $p(x, \omega), (x, \omega) \in \Omega \times I$ such that:

\[
\begin{align*}
\Delta p + k^2 p &= 0 \quad \text{over } \Omega \times I \\
p &= p_d \quad \text{over } \partial_p \Omega \times I \\
L_v(p) &= v_d \quad \text{over } \partial_v \Omega \times I \\
p - ZL_v(p) &= h_d \quad \text{over } \partial_Z \Omega \times I
\end{align*}
\]

(1)

where $k = (1 - i \eta) \frac{\omega}{c_0}$ is the wave number ($\eta$ is the absorption coefficient), $p_d$ a prescribed pressure, $v_d$ a prescribed velocity, $Z$ a given impedance and $h_d$ a given function. Operator $L_v(\square)$ is defined by: $L_v(\square) = \frac{i}{\rho_0 \omega} \partial_n \nabla (\square) = \frac{i}{\rho_0 \omega} n^T \nabla (\square)$ ($n$ is the outward normal). $\partial_p \Omega$, $\partial_v \Omega$ and $\partial_Z \Omega$ are the parts of the boundary $\partial \Omega$ of $\Omega$ where the pressure, the velocity and a Robin condition are respectively prescribed. The uniqueness of the solution of this reference problem is ensured by a strictly positive $\eta$. If $\Omega$ is partitioned into $n_{el}$ non-overlapping elements $\Omega_e$,
the continuity additional equations must also be verified along $\Gamma_{e,e'} = \Omega_e \cap \Omega_{e'}$:

$$
\begin{align*}
  p_e - p_{e'} &= 0 \quad \text{along } \Gamma_{e,e'} \times I \\
  L_v(p_e) + L_v(p_{e'}) &= 0 \quad \text{along } \Gamma_{e,e'} \times I
\end{align*}
$$

(2)

3. THE VTCR VARIATIONAL FORMULATION OF THE REFERENCE PROBLEM

Lest us define the functional space of the functions which satisfy the homogeneous Helmholtz governing equation, i.e. the first equation in (1):

$$
S^e = \left\{ p_e; \Delta p_e + k^2 p_e = 0 \text{ over } \Omega_e \times I \right\}
$$

(3)

For the 2D case, these functions can be represented by waves propagating in all possible 2D directions (see [12]):

$$
p_e(x, \omega) \in S^e \iff p_e(x, \omega) = \int_{\theta=0}^{2\pi} X_e(\theta, \omega)e^{ik_e(\theta,\omega)x}d\theta
$$

(4)

where $k_e = k(\cos \theta, \sin \theta)$. $X_e(\theta, \omega)$ (which correspond to the amplitudes of the waves propagating in $\Omega_e$) are the unknowns of the problem. According to the VTCR (see [1] or [13] for acoustics), Problem (1),(2) can be formulated in: find $(p_1, \ldots, p_e, \ldots, p_{n_e}) \in S^1 \times S^e \times \cdots \times S^{n_e}$ such that:

$$
\begin{align*}
  \sum_{\Omega_e} \mathfrak{R}\left\{ \int_{\partial\Omega_e} (p_e - p_{d,e}) L_v[\delta p_e]ds + \int_{\partial\Omega_e} (L_v[p_e] - v_{d,e}) \delta p_e ds \right. \\
  + \frac{1}{2} \int_{\partial\Omega_e} \left[ (L_v[p_e] - 1)[L_v[p_e]] + (L_v[p_e] + h_{d,e}[Z_e])\delta p_e \right] ds \\
  \left. + \sum_{\Gamma_{e,e'}} \frac{1}{2} \int_{\Gamma_{e,e'}} \left[ (p_e - p_{e'}) L_v[\delta p_e - \delta p_{e'}] + L_v[p_e + p_{e'}](\delta p_e + \delta p_{e'}) \right] ds \right\} = 0
\end{align*}
$$

(5)

where $p_{d,e}$, $v_{d,e}$ and $h_{d,e}$ are the data we introduced in last section restricted to the boundary of $\Omega_e$. $\mathfrak{R}\{\square\}$ is the real part of the quantity $\square$ and $\overline{\square}$ represents the complex conjugate value of the quantity $\square$.

As a consequence, all that is necessary in order to develop a VTCR approximation of Problem (1),(2) is to verify (5) in a finite-dimension subspace $S^{e,h}$ of $S^e$. Here, we chose a Fourier series expansion of the wave amplitudes limited to the first $2N_e + 1$ terms (see [13]):

$$
X_e(\theta, \omega) = \sum_{n=-N_e}^{N_e} X^n_e(\omega)e^{in\theta}
$$

(6)

where $N_e$ is sufficiently large to ensure a good approximation (see [13]: $N_e$ is based on the energy of the shape function defined in (7)). In other words, the approximation is spanned by functions $\Phi^n_e(x, \omega)$:

$$
p_e(x, \omega) \approx \sum_{n=-N_e}^{N_e} X^n_e(\omega)\Phi^n_e(x, \omega)
$$

(7)

where $\Phi^n_e(x, \omega) = \int_{\theta=0}^{2\pi} e^{in\theta}e^{ik_e(\theta,\omega)x}d\theta$. 
For a fixed frequency $\omega$, $X^p(\omega)$ are the VTCR unknown DOFs of Problem (1),(2), and are related to the amplitudes of the waves which propagate in $\Omega_e$. The substitution of (7) into (5) leads to the matrix system:

$$K(\omega)X(\omega) = F(\omega)$$

(8)

where $K$ and $F$ are respectively the projections of the bilinear and linear forms of (5) onto the space generated by functions $\Phi^p$. Let us denote $N$ the size of $X$.

4. THE COMBINING OF PGD AND VTCR TO SOLVE FREQUENCY BAND PROBLEMS

As explained before, the innovative aspect of our approach is that it uses the PGD separated representation to solve Problem (8) in the frequency range $I = \left[\omega_0 - \frac{\Delta \omega}{2}, \omega_0 + \frac{\Delta \omega}{2}\right]$. As a consequence, an good approximation of $X(\omega)$ of (8) in searched the form:

$$X(\omega) \simeq X^M(\omega) = \sum_{m=1}^{M} X_m \lambda_m(\omega)$$

(9)

where $X^M(\omega)$ (like $X(\omega)$) is a vector defined on $C^N \otimes T$ ($T$ being the space of frequency dependent functions whose square integration on $I$ is finite), $X_m$ are vectors of $C^N$ and $\lambda_m(\omega)$ functions of $T$. As $X^M(\omega)$ is in relation to the wave propagation directions, the proposed separation of the variables is done between the angular propagation direction and the frequency.

None of these functions is known a priori. Then, the question is to find the optimal decomposition and to calculate its terms. Different algorithms exist in the literature to calculate $X_m$ and $\lambda_m(\omega)$ (see [14] for example). In our case, matrix $K(\omega)$ (see (8)) has no symmetry. Therefore, among all the possible PGD algorithms, our natural choice was the Petrov-Galerkin-based PGD algorithm, which requires no artificial symmetrization of the problem.

This algorithm however requires the definition of the variational formulation of the matrix problem (8), which can be expressed as: find $X(\omega)$ such that

$$K(X(\omega), Y(\omega)) = F(Y(\omega)) \quad \forall Y(\omega) \in C^N \otimes T$$

(10)

where $K(X(\omega), Y(\omega)) = \int_{\omega_0-\Delta \omega/2}^{\omega_0+\Delta \omega/2} Y(\omega)^T K(\omega) X(\omega)d\omega$ and $F(Y) = \int_{\omega_0-\Delta \omega/2}^{\omega_0+\Delta \omega/2} Y(\omega)^T F(\omega)d\omega$.

In order to develop the algorithm, let us assume that $X_m$ and $\lambda_m(\omega)$ are known for $m = 1...M - 1$ and that we are now seeking the enrichment pair $(X_M, \lambda_M)$. The Petrov-Galerkin-based PGD algorithm for calculating these terms consists in using the following two orthogonality criteria:

$$K(X^{M-1} + X_M \lambda_M(\omega), Y' \gamma(\omega)) = F(Y' \gamma(\omega)) \quad \forall Y' \in C^N$$

(11)

$$K(X^{M-1} + X_M \lambda_M(\omega), Y' \gamma(\omega)) = F(Y' \gamma(\omega)) \quad \forall Y' \in T$$

(12)

$Y \in C^N$ and $\gamma(\omega) \in T$ correspond to another pair of a constant vector and a frequency-dependent function.

Of course, additional equations must be added to (11) and (12) in order to define functions $(Y, \gamma)$. We use the following equations:

$$K(X' \lambda_M(\omega), Y \gamma(\omega)) = << X' \lambda_M(\omega), X_M \lambda_M(\omega) >> \quad \forall X' \in C^N$$

(13)

$$K(X_M' \lambda(\omega), Y \gamma(\omega)) = << X_M' \lambda(\omega), X_M \lambda_M(\omega) >> \quad \forall \lambda' \in T$$

(14)

$<< \ldots >>$ corresponds to the inner product defined by

$$<< X \lambda(\omega), Y \gamma(\omega) >> = \int_{\omega_0-\Delta \omega/2}^{\omega_0+\Delta \omega/2} \gamma(\omega) Y^T H(\omega) X \lambda(\omega)d\omega$$

with $H(\omega) = \tilde{H}h(\omega)$, where $\tilde{H}$ is a...
constant matrix equal to the mean value of matrix $K(\omega)$ over the frequency band, and $\tilde{h}(\omega)$ is a frequency-dependent function which corresponds to the mean value of the coefficients of the diagonal of $K(\omega)$. With that particular choice, which is in relation with the physics of the problem, the following separation property holds:

$$
<< X_\lambda(\omega), Y_\gamma(\omega) >> = \left( X^T \tilde{H} Y \right) \int_{\omega_0-\Delta_\omega/2}^{\omega_0+\Delta_\omega/2} \lambda(\omega)\gamma(\omega) d\omega
$$

(15)

This property is useful in terms of convergence demonstration (see [14]).

Consequently, the use of the algorithm consists in deriving simultaneously $(X_M, \lambda_M)$ and $(Y, \gamma)$ which verify Equations (11), (12), (13) and (14). In order to do that we use a power-type iteration, which means that these equations are solved iteratively until convergence, i.e. until each function has reached a fixed value (see [14]). In order to test the convergence of these iterations, we use the following criterion:

$$
\int_{\omega_0-\Delta_\omega/2}^{\omega_0+\Delta_\omega/2} (X_M^{(q)} - X_M^{(q-1)}) (\omega) \left( X_M^{(q)} - X_M^{(q-1)}(\omega) \right) d\omega < \delta_q
$$

(16)

where $(X_M^{(q)} - X_M^{(q-1)})$ (resp. $(X_M^{(q-1)} - X_M^{(q-1-1)})$) is the pair $(X_M, \lambda_M)$ calculated at the iteration with the index $(q)$ (resp. $(q-1)$) and $\delta_q$ the desired accuracy.

As soon as $(X_M, \lambda_M)$ has converged, the procedure is repeated until the convergence of the global enrichment procedure $X(\omega) = X^M(\omega) = \sum_{m=1}^{M} X_m \lambda_m(\omega)$. The stopping criterion used is:

$$
\epsilon_M(X^M) = \int_{\omega_0-\Delta_\omega/2}^{\omega_0+\Delta_\omega/2} \frac{(K(\omega)X^M(\omega) - F(\omega))^T(K(\omega)X^M(\omega) - F(\omega))}{\int_{\omega_0-\Delta_\omega/2}^{\omega_0+\Delta_\omega/2} F(\omega)^T F(\omega) d\omega} < \delta_M
$$

(17)

where $\delta_M$ is related to the accuracy of the PGD separated representation of the solution of Problem (8).

The initialization values of this algorithm are

$$
\lambda^{(0)}_M(\omega) = \gamma^{(0)}_M(\omega) = \sqrt{(K(\omega)X^{M-1}(\omega) - F(\omega))^T(K(\omega)X^{M-1}(\omega) - F(\omega))}
$$

(18)

which make a link between the data and the physics of the problem.

5. NUMERICAL EXAMPLE

In order to test the proposed approach, we studied the 2-D acoustic car cavity whose geometry is defined on Figure 1. The cavity is filled with air, then the physical data involved for defining $k$ and $L_c$ introduced in Section 2 are $\rho_0 = 1, 25 \text{ kg/m}^3$, $c_0 = 340 \text{ m/s}$, $h = 0, 0005$. A boundary condition with a prescribed velocity with $v_d = 1 \text{ m/s}$ or $v_d = 0 \text{ m/s}$ has been used on the windows, the ceil and the roof of the car (see Figure 1 for their localization). A Robin-type boundary condition has been used on the seats, with $Z = 10 - 5i \text{ Pa.s/m}$. The acoustic car cavity excitation hence comes from the prescribed velocity with $v_d = 1 \text{ m/s}$. The central frequency of the frequency range $I$ (see Section 4 for its definition) is $\omega_0 = 2\pi \times 2, 500 \text{ rad/s}$ and the bandwidth is $\Delta_\omega = 2\pi \times 1000 \text{ rad/s}$.

For the computations, the whole domain has been subdivided in 8 sub-domains of different shapes. Figure 1 shows them (the straight lines in the acoustic cavities correspond to the boundaries of the sub-domains). In each sub-domain $e$, we have used $2N_e + 1$ shape functions
Figure 1: Top: definition of the numerical problem of a 2-D acoustic car cavity used in Section 5 to test the efficiency of the strategy described in Section 4. Bottom: real part of the pressure of the response of the considered problem at $\omega = 2\pi \times 2,500$ rad/s.

The reference solution has been computed thanks to the VTCR, frequency by frequency, every 1 Hz. The result of such a VTCR computation at 2500 Hz is depicted in Figure 1. We can see that the continuity across the sub-domains (which is the only equation that is approximated in the problem defined in (1)) is satisfied. Then the solution seems to be sufficiently reliable to represent correctly the exact solution. Moreover, the energy of the solution in a measure area localized near the head of the driver (see the black squared area in Figure 1) has been computed, frequency by frequency, every 1 Hz. The result of such a computation can be seen in the black curve of Figure 2. As one can see, this curve is very sharp, presents many oscillations and vibrational peaks, and seems to be a response with a frequency-dependant enough complexity to be a good challenge in order to test the efficiency of the VTCR-PGD strategy presented in Section 4.

The PGD algorithm presented in Section 4 has been used to predict the reference solution. As explained before, the reference solution considered here was the solution given by the VTCR, frequency by frequency. This reference solution has been retained as we just wanted to test the efficiency of the VTCR-PGD algorithm. Another reference solution, such as a FEM computation, would introduce another error due to the use of another description of the problem (a FEM polynomial approximation versus a VTCR wave approximation), which is a question that is out of the context of this study. For the VTCR-PGD algorithm described in Section 4, the parameter values $\delta_q = 0.0001$ (see (16)) and $\delta_M = 0.0001$ (see (17)) have been used. These small values have been motivated by the fact that we wanted to observe the convergence of the algorithm. Other values can be selected to get a better speed of convergence.
of the algorithm (values bigger than the one selected here) with almost no degradation in the quality of the converged solution, but this will be considered in future works and is not in the scope of this study.

The result obtained with the VTCR-PGD algorithm is the red curve depicted in Figure 2. As said before, the black curve corresponds to the reference curve, computed frequency by frequency. As one can see, the two curves are rather similar. Almost all the peaks are recovered, whatever their sharpness behavior, on the whole frequency band. Even the highest peak near 2740 Hz is recovered. This illustrate the efficiency of the proposed method.

![Image of Figure 2](image)

**Figure 2**: Mean energy [J] in the measure area of the problem considered in Figure 1. The black curve corresponds to the reference curve and the red curve corresponds to the VTCR-PGD approximated curve.

6. CONCLUSIONS

We proposed a new version of the VTCR for the resolution of medium-frequency problems in frequency bands. It is based on its combination with the PGD. It leads to a decomposition of the solution into a basis of constant vectors and a set of frequency-dependent functions. The proposed decomposition is obtained using a power-type algorithm. A numerical example showed the effectiveness of the method and the excellent convergence properties of the algorithm. After these promising results, the method will be extended to more complex examples which, along with the definition of new algorithms and the comparison with the established calculation tools, will be the topics of our future research on the subject.

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