CALCULATING THE FORCED RESPONSE OF CYLINDERS USING THE WAVE AND FINITE ELEMENT METHOD

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ABSTRACT

The dynamic response of cylinders can be obtained analytically in very few (and simple) cases. For complicated (thick or anisotropic) cylinders, researchers often resort to the finite element (FE) method which can be computationally expensive at higher frequencies. In this paper, the response of cylinders is obtained using the wave and finite element (WFE) technique. The FE model of a small rectangular segment of the cylinder can be post-processed using periodic structure theory to yield its wave characteristics. Thus, cylinders with arbitrary complexity can be considered since the full power of FE methods can be utilised to obtain the FE model of the small segment. Then, the response of the cylinder is posed as an inverse Fourier transform. However, since there are an integer number of wavelengths around the circumference of a cylinder, one of the integrals in the inverse Fourier transform reduces to a simple summation, whereas the other is resolved analytically using contour integration and the residue theorem. The result is a computationally efficient technique for obtaining the response of cylinders to arbitrarily distributed loads.

1. INTRODUCTION

Many engineering structures are cylindrical, e.g. pipes, acoustic ducts, etc. Modelling the vibrational behaviour of such structure is important for many applications such as sound transmission and radiation, disturbance propagation, shock response, fatigue analysis, etc.

Current engineering structures are increasingly made from lightweight, fibre-reinforced constructions. In this case, developing comprehensive mathematical models which accurately describe the stress-strain distribution through the thickness is extremely difficult at best. Researchers attempted to use the three dimensional theory of elasticity to obtain the dispersion relations of cylindrical shells [1-3]. However, due to the algebraic complexity involved in such development, various approximations and assumptions have to be introduced [4] and these efforts yield solutions for thin cylindrical shells only [5-7], with purely real dispersion curves.
Consequently, to characterise the dynamics of cylinders, researchers often resort to numerical techniques such as the finite element (FE) method. As for the response of the structure, it can be obtained using standard FE; this will lead to impractically large models at higher frequencies. As for obtaining the wave characteristics, a thorough literature review shows that Nelson et al. [8] were the first to analyse wave propagation in laminated orthotropic cylinders. This was followed by the work of Huang and Dong [9] who used a similar approach to study anisotropic cylinders. Mahapatra and Gopalakrishnan [10] used the spectral FE approach [11] to analyse wave propagation in uniform composite thin cylinders. Another FE approach, the wave and finite element (WFE) method, was used to model the wave behaviour of cylinders [12-14]. The WFE approach was used for modelling the wave behaviour of one- [12, 15-17] and two-dimensional structures [18] and cylinders [19, 20].

Researches have used a one-dimensional WFE model to find the response of cylinders to a point force, [14], i.e., they treated the cylinder as a waveguide rather than a two-dimensional structure. In this paper, the cylinder’s axisymmetry is exploited as in [19], efficiently formulate the forced response of the cylinder to general loading. The benefit of this approach is two folds: first, the size of the WFE model is relatively very small allowing the a priori characterisation of the circumferential waves in the circumferential direction and second, the knowledge of the waves is efficiently used to resolve the response analytically.

The approach starts with a standard FE model of a small rectangular segment of the cylinder and the WFE developments of axisymmetric structure are used [19]. Then, the response of the cylinder to a convected harmonic pressure (CHP) is formulated. This formulation allows representing the response of the cylinder to an arbitrarily distributed (time harmonic) load as a linear combination of the response of the cylinder to CHPs. This is possible since the arbitrarily distributed load can be expressed as a linear combination of CHPs via Fourier transform. Thus, the problem of finding the response of the cylinder reduces to an inverse Fourier transform which can be resolved analytically using contour integration and the residue theorem. The approach is similar to that described by Renno and Mace [21] for obtaining the response of infinite, two-dimensional media. However, here, the axisymmetry is exploited in two ways. First, one of the integrals in the inverse Fourier transform becomes a simple summation, rather than a numerical integral. Second, the one dimensionalisation of the problem (by decomposing the response into a sum of circumferential harmonics) allows obtaining the response of finite cylinders, rather than infinite structures only. The approach is numerically efficient, and can handle arbitrarily complicated cylinders excited by arbitrarily distributed loads.

The remaining of this paper is organised as follows. The implementation of the WFE method for axisymmetric structures is reviewed in Section 2. The response of infinite cylinders to general loads is developed in Section 3, and then the response of finite cylinders is treated in Section 4. Several numerical examples are presented in Section 5 and conclusions for this work are drawn in Section 6.

2. WFE METHOD

In this section, the WFE method for curved axisymmetric structures is briefly reviewed [19]. A cylinder of mean radius of curvature $R$ and thickness $h$, Figure 1a, can be described through cylindrical coordinates $y$, $r$, and $\alpha$. A time harmonic disturbance, at a frequency $\omega$, of the mid-surface of this structure propagates in a helical pattern as

$$w(r, \alpha, y, t) = W(r) \exp(-ik_\alpha \alpha) \exp(-ik_y y) \exp(i\omega t),$$  \hspace{1cm} (1)

where $W(r)$ is the complex wave amplitude and $k_\alpha$ and $k_y$ are the projections of the physical wavenumber in the circumferential and axial directions, respectively.
Since the structure is homogenous around the circumference and along the axis, a curved segment, Figure 1b, can be used as a unit cell from which the whole structure can be obtained by tessellation. It is convenient to define an alternative circumferential coordinate \( x = R\alpha \). The latter definition implies that the projection of the physical wavenumber on the \( x \)– axis is \( k_x = k_r / R \). If the periodic angle \( \Delta \alpha \) is small, then the curved segment of Figure 1b can be approximated using a flat rectangular segment with dimensions \( \Delta x \times \Delta y \) with \( \Delta x = R \Delta \alpha \).

![Figure 1 (a) Axisymmetric structure and cylindrical coordinates; (b) small curved segment of the axisymmetric structure.](image)

To apply the WFE approach, the FE model of this segment is obtained using any FE package, and the segment is meshed using any number of elements. The only restriction is that the nodes at each side of the segment are arranged identically. This can be done using shell elements, or using a stack of solid elements, Figure 2a. In all cases, the vector of degrees of freedom (DOFs), \( \mathbf{q} \), of the segment (illustrated in Figure 2b) is partitioned as

\[
\mathbf{q}^T = \begin{bmatrix} \mathbf{q}_{lb}^T & \mathbf{q}_{rb}^T & \mathbf{q}_b^T & \mathbf{q}_{rt}^T & \mathbf{q}_i^T & \mathbf{q}_l^T & \mathbf{q}_r^T \end{bmatrix},
\]

where the superscript \( T \) denotes transposition and the subscripts \( l, r, b, t, i \) correspond to left, right, bottom, top and internal nodes. The vector of internal nodal forces, \( \mathbf{f} \), and external nodal forces, \( \mathbf{e} \), can be partitioned in the same manner.

To include the effect of the curvature of the structure, the local coordinates (where the FE model was obtained) shall be rotated around the \( y \)– axis by an angle \( \Delta \alpha \). Thus, the FE matrices of the “curved” segment become

\[
\mathbf{M} = \mathbf{\bar{R}}^T \mathbf{M}_{loc} \mathbf{\bar{R}}, \quad \mathbf{C} = \mathbf{\bar{R}}^T \mathbf{C}_{loc} \mathbf{\bar{R}}, \quad \mathbf{K} = \mathbf{\bar{R}}^T \mathbf{K}_{loc} \mathbf{\bar{R}},
\]

where \( \mathbf{M}_{loc}, \mathbf{C}_{loc} \) and \( \mathbf{K}_{loc} \) are the segment’s mass, viscous damping and stiffness matrices in the local \( (xyz) \) coordinate system of Figure 2a and \( \mathbf{\bar{R}} \) is the rotation matrix given as

\[
\mathbf{\bar{R}} = \text{diag}(\mathbf{I} \mathbf{R}_{rb} \mathbf{I} \mathbf{R}_{rt} \mathbf{I} \mathbf{R}_r \mathbf{I} \mathbf{I} \mathbf{I}),
\]

where \( \text{diag}(.) \) here represents a block diagonal matrix, and \( \mathbf{R}_{rb}, \mathbf{R}_{rt} \) and \( \mathbf{R}_r \) allow the rotation around the \( y \)– axis of the \( (rb), (rt) \) and \( (r) \) nodes of the segment, Figure 2b. These rotation matrices are block diagonal, and comprise unitary sub-matrices \( \mathbf{r} \) defined as

\[
\mathbf{r} = \begin{bmatrix} \cos(\Delta \alpha) & 0 & -\sin(\Delta \alpha) \\ 0 & 1 & 0 \\ \sin(\Delta \alpha) & 0 & \cos(\Delta \alpha) \end{bmatrix}.
\]
Figure 2 (a) FE mesh of a small rectangular prismatic segment of the axisymmetric structure; (b) node numbering.

For time harmonic motion at frequency $\omega$, the governing equation of the segment of Figure 2a is

$$
\begin{bmatrix}
K + i\omega C - \omega^2 M
\end{bmatrix} \mathbf{q} = \mathbf{f} + \mathbf{e}.
$$

(6)

The structure can now be treated as a two-dimensional periodic structure. Under the free passage of a wave of the form $\exp(-ik_x x) = \exp(-i\kappa_x x)$ in the circumferential direction, the DOFs of Equation (2) are related by

$$
\mathbf{q} = \Lambda_q \mathbf{q}_{\text{red}}, \quad \Lambda_q = \begin{bmatrix}
I & 0 & 0 & 0 & 0 & 0 \\
\lambda_x I & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & I & 0 & 0 & 0 \\
0 & 0 & 0 & \lambda_x I & 0 & 0 \\
0 & 0 & 0 & 0 & I & 0 \\
0 & 0 & 0 & 0 & 0 & I
\end{bmatrix}, \quad \mathbf{q}_{\text{red}} = \begin{bmatrix}
\mathbf{q}_{lb} \\
\mathbf{q}_b \\
\mathbf{q}_{rb} \\
\mathbf{q}_r \\
\mathbf{q}_{lt} \\
\mathbf{q}_t \\
\mathbf{q}_{rt} \\
\mathbf{q}_i
\end{bmatrix},
$$

(7)

and $\lambda_x = \lambda_x = \exp(-ik_x \Delta x) = \exp(-i\kappa_x \Delta x)$. Since closed cylinders are the topic of this paper, it is worth mentioning at this point that the circumferential wavenumber can take on integer values only, i.e., $k_x = 0, 1, 2, \cdots$. Using the reduced vector of DOFs defined in Equation (7), Equation (6) can be written as

$$
\Lambda_q^H \begin{bmatrix}
K + i\omega C - \omega^2 M
\end{bmatrix} \Lambda_q \mathbf{q}_{\text{red}} = \Lambda_q^H \mathbf{f},
$$

(8)

where the superscript $H$ denotes complex transposition. The right hand side of the above equation can be expanded to yield

$$
\Lambda_q^H \mathbf{f} = \begin{bmatrix}
\{f_{lb} + \lambda_x^{-1} f_{rt}\}^T \\
\{f_b + \lambda_x^{-1} f_{rb}\}^T \\
\{f_{rb} + \lambda_x^{-1} f_{rt}\}^T \\
\{f_r + \lambda_x^{-1} f_{rb}\}^T \\
\{f_{rt}\}^T
\end{bmatrix},
$$

(9)
where one notes that there are no forces acting on the internal nodes, and thus \( f_i = 0 \). Furthermore, the equilibrium conditions at the right edge of the segment imply that \( f_f + \lambda^{-1} f_r = 0 \). Now, the bottom and top DOFs and internal forces can be grouped as

\[
q_B = \{ q_{ib} \}, q_T = \{ q_{ib} \}, q_o = \{ q_i \}, f_B = \{ f_{ib} + \lambda^{-1} f_{ib} \}, f_T = \{ f_{ib} + \lambda^{-1} f_{ib} \}.
\]  

Equation (8) is now rearranged into

\[
\begin{bmatrix}
\tilde{D}_{bb} & \tilde{D}_{bt} & \tilde{D}_{bo} \\
\tilde{D}_{tb} & \tilde{D}_{tt} & \tilde{D}_{to} \\
\tilde{D}_{ob} & \tilde{D}_{ot} & \tilde{D}_{oo}
\end{bmatrix}
\begin{bmatrix}
q_B \\
q_T \\
q_o
\end{bmatrix}
= \begin{bmatrix}
f_B \\
f_T \\
0
\end{bmatrix},
\]

where \( \tilde{D} = \Lambda^H \left[ K + i\omega C - \omega^2 M \right] \Lambda_q \). Finally, the DOFs of the left edge and internal nodes (concatenated in \( q_o \)) are eliminated from Equation (11), yielding

\[
\begin{bmatrix}
D_{bb} & D_{bt} \\
D_{tb} & D_{tt}
\end{bmatrix}
\begin{bmatrix}
q_B \\
q_T
\end{bmatrix}
= \begin{bmatrix}
f_B \\
f_T
\end{bmatrix},
\]

where the partitions of the dynamic stiffness matrix \( D \) are related to those of \( \tilde{D} \). The size of the system in Equation (12) is \( 2n \times 2n \), where \( n \) is the number of DOFs of the bottom left corner and bottom edge nodes of the segment.

At a given frequency \( \omega \), and for a given value of \( k_x \) or \( k_y \), waves in the \( y \)-direction will be of the form \( \exp(-ik_y y) \) where \( k_y \) can be real, imaginary or complex. The admissible values of \( \lambda_y = \exp(-ik_y \Delta y) \) can be found by imposing periodicity and equilibrium conditions between the bottom and top nodes of the segment leading to an eigenvalue problem[16].

The solution of the eigenproblem yields the propagation constants \( \lambda^+_y \) and the corresponding wavenumbers \( k^+_y \) (\( j = 1, \ldots, 2n \)). For complicated media with many DOFs, care needs to be taken when solving the eigenvalue problem as various numerical problems might arise when using the transfer matrix form [16].

It has been shown [22, 23] that the eigenvalues of the transfer matrix occur in reciprocal pairs as \( \lambda^+_y \) and \( \lambda^-_y = 1/\lambda^+_y \) with wavenumbers \( k^+_y \) and \( k^-_y = -k^+_y \) (\( j = 1, \ldots, n \)), corresponding to pairs of positive- and negative-going waves respectively. Associated with these eigenvalues are the positive- and negative-going eigenvectors \( \varphi^+_y \) and \( \varphi^-_y \), respectively, which are referred to as the wavemodes. Every wavemode can be partitioned into a sub-vector of DOFs and internal forces as

\[
\varphi_j = \left[ \begin{array}{c} \varphi^+_j \\ \varphi^-_j \end{array} \right]^T.
\]  

With the positive- and negative-going waves identified, the wavemodes are grouped as

\[
\Phi^+ = \left[ \varphi^+_1 \cdots \varphi^+_n \right], \quad \Phi^- = \left[ \varphi^-_1 \cdots \varphi^-_n \right], \quad \Phi = \left[ \Phi^+ \Phi^- \right].
\]

The left eigenvectors of the transfer matrix can be obtained and are partitioned as
\[ \Psi_j = \begin{bmatrix} \Psi_f & \Psi_q \end{bmatrix}, \]  
\text{(15)}

and further grouped into the matrix

\[ \Psi = \begin{bmatrix} \Psi_f \end{bmatrix}^T \begin{bmatrix} \Psi_q \end{bmatrix}^T. \]
\text{(16)}

The left and right wavemodes are orthogonal, and can be normalised so that

\[ \Psi^\dagger \Phi^\dagger = I \quad \text{and} \quad \Psi^T \Phi = \text{diag} \left( \lambda_j \right). \]
\text{(17)}

The partitions of the left and right wavemodes can be used to form the matrices

\[ \Phi_q^+ = \begin{bmatrix} \phi_{q,1}^+ & \cdots & \phi_{q,n}^+ \end{bmatrix}, \quad \Psi_q^+ = \begin{bmatrix} \psi_{q,1}^+ \end{bmatrix}^T \cdots \begin{bmatrix} \psi_{q,n}^+ \end{bmatrix}^T. \]
\text{(18)}

Similar expressions hold for \( \Phi_q^-, \Psi_q^-, \Phi_f^+ \) and \( \Phi_f^- \). These matrices define a transformation between the physical and the wave domains [24]

\[ \begin{bmatrix} \mathbf{q}_B \\ \mathbf{f}_B \end{bmatrix} = \Phi \mathbf{a}, \quad \mathbf{a} = \begin{bmatrix} \mathbf{a}^+ \\ \mathbf{a}^- \end{bmatrix}. \]
\text{(19)}

where \( \mathbf{a}^+ \) and \( \mathbf{a}^- \) are the amplitudes of the waves travelling in the positive and negative directions, respectively.

In practice, as in modal analysis, only \( m \) pairs of (positive-and negative-going) waves might be retained, so that \( \Phi_{q,f}^\pm \) and \( \Psi_{q,f}^\pm \) are \( (n \times m) \) and \( (m \times n) \) matrices, respectively. The number retained can be different at different frequencies. All the propagating waves, for which \( |\lambda_j| = 1 \), must be retained together with the least-rapidly attenuating waves, i.e., for \( |\lambda_j| < 1 \), all those for which \( |\lambda_j| \) is greater than some user-defined value. The reasons for reducing the size of the wave basis are partly that the size of the model will be smaller, but primarily that the calculation of the high-order wavemodes, which decay very rapidly with distance (by orders of magnitude over \( \Delta y \)), is very prone to poor numerical conditioning [16].

In this section, the wave characteristics of a curved shell were obtained using the FE model of a small segment only through the exploitation of the axisymmetry. Next, the response to a CHP and later to a general load will be treated.

3. RESPONSE TO GENERAL EXCITATION

The response of a cylinder to a (time harmonic) generally distributed load will now be formulated. The loading can be a vectorial quantity with components acting along any direction and the only restriction imposed is that every component is separable and can be written in the form \( p(\alpha, y, r) = g(\alpha, y) h(r) \). For instance, the structure in Figure 2a can be excited along the three axes. In this case, the loading will be

\[ p(\alpha, y, z, t) = \begin{cases} p_x(\alpha, y, r, t) & \text{for} \, g_y(\alpha, y) h_y(r) \exp(\text{i} \omega t), \\ p_y(\alpha, y, r, t) & \text{for} \, g_x(\alpha, y) h_x(r), \\ p_z(\alpha, y, r, t) & \text{for} \, g_x(\alpha, y) h_x(r) \exp(\text{i} \omega t). \end{cases} \]
\text{(20)}
where the subscripts $\alpha$, $y$ and $r$ indicate the direction of the components of $p$. The developments presented below are for one component. Since all the components of the loading will be later resolved into consistent nodal forces, the contributions of additional components can be simply added to the consistent nodal forces.

3.1 Response to unit amplitude convected harmonic pressure (CHP)

In two dimensions, the CHP can be expressed as

$$p_{\text{CHP}}(\alpha, y, r) = e^{-i\gamma_{\alpha} \Delta \alpha} e^{-i\gamma_{y} \Delta y} h(r).$$  \hfill (21)

Under this forcing, the propagation constants of the response of the medium are

$$\mu_{x} = \mu_{\alpha} = e^{-i\gamma_{\alpha} \Delta \alpha} \quad \text{and} \quad \mu_{y} = e^{-i\gamma_{y} \Delta y},$$  \hfill (22)

and are imposed by the CHP. Consistent nodal forces can be obtained straightforwardly [25]

$$e = \tilde{R} \int_{\Omega} \left[ p_{\text{CHP}}(\alpha, y, r) N^{T} (R\alpha, y, r) \right] d\Omega,$$  \hfill (23)

where $\Omega$ is the domain of the curved segment (i.e., $\Delta \alpha \times \Delta y$), $N$ is a vector of shape functions and the rotation matrix $\tilde{R}$ indicates that the external consistent nodal forces should be represented in the rotated system as a reflection of the curvature of the cylinder. The external nodal forces are partitioned in the same way as the DOFs and internal nodal forces (Section 2) and thus the equations of motion are written as

$$\begin{bmatrix} \tilde{D}_{BB} & \tilde{D}_{BT} & \tilde{D}_{BO} \\ \tilde{D}_{TB} & \tilde{D}_{TT} & \tilde{D}_{TO} \\ \tilde{D}_{OT} & \tilde{D}_{OT} & \tilde{D}_{OO} \end{bmatrix} \begin{bmatrix} q_{B} \\ q_{T} \\ q_{O} \end{bmatrix} = \begin{bmatrix} f_{B} \\ f_{T} + \tilde{e}_{T} \\ 0 \end{bmatrix} + \begin{bmatrix} \tilde{e}_{B} \\ \tilde{e}_{T} \\ \tilde{e}_{O} \end{bmatrix},$$  \hfill (24)

where $\tilde{e} = \Lambda_{q} (\mu_{y})^{H} e$ and

$$\tilde{e}_{B} = \begin{bmatrix} e_{b} + \mu_{y}^{*} e_{ib} \\ e_{i} \end{bmatrix}, \quad \tilde{e}_{T} = \begin{bmatrix} e_{T} + \mu_{y}^{*} e_{iT} \\ e_{l} \end{bmatrix} \quad \text{and} \quad \tilde{e}_{O} = \begin{bmatrix} e_{O} + \mu_{y}^{*} e_{lO} \\ e_{i} \end{bmatrix}.$$  \hfill (25)

The internal, bottom and top edge DOFs (augmented in $q_{O}$) can be eliminated yielding

$$\begin{bmatrix} D_{BB} & D_{BT} \\ D_{TB} & D_{TT} \end{bmatrix} \begin{bmatrix} q_{B} \\ q_{T} \end{bmatrix} = \begin{bmatrix} f_{B} \\ f_{T} + f_{T} \end{bmatrix} + \begin{bmatrix} e_{B} \\ e_{T} \end{bmatrix}.$$  \hfill (26)

Under the CHP loading, the periodicity and equilibrium conditions are

$$q_{T} = \mu_{y} q_{B} \quad \text{and} \quad \mu_{y} f_{B} + f_{T} = 0.$$  \hfill (27)

Equation (26) is rearranged into the transfer matrix form as

$$\mu_{y} \begin{bmatrix} q_{B} \\ f_{B} \end{bmatrix} = T \begin{bmatrix} q_{B} \\ f_{B} \end{bmatrix} + \begin{bmatrix} D_{BT}^{-1} \\ -D_{TT} D_{BT}^{-1} I \end{bmatrix} \begin{bmatrix} e_{B} \\ e_{T} \end{bmatrix}.$$  \hfill (28)
Hence,
\[
\left[ \mu_y I - T \right] \begin{bmatrix} q_B \\ f_B \end{bmatrix} = \begin{bmatrix} D_{BT}^{-1} & 0 \\ -D_{TT} D_{BT}^{-1} & I \end{bmatrix} \begin{bmatrix} e_B \\ e_T \end{bmatrix}.
\]  \tag{29}

Equation (29) can be solved by inverting the matrix on the left hand side. However, it is preferable to solve it by transforming the DOFs and internal forces into the wave domain. In addition to avoiding possible numerical problems, this primarily allows the knowledge of the characteristics of free wave propagation to be exploited when analyzing the response of the cylinder to general excitations. Equation (19) is substituted into Equation (29) which is then premultiplied by \( \Psi \), and the orthonormality properties of Equation (17) are used to yield
\[
\left[ \mu_y I - \text{diag}\left( \lambda_y^i \right) \right] a = \Psi \begin{bmatrix} D_{BT}^{-1} & 0 \\ -D_{TT} D_{BT}^{-1} & I \end{bmatrix} \begin{bmatrix} e_B \\ e_T \end{bmatrix}.
\]  \tag{30}

The matrix on the left hand side of the above equation can be easily inverted to obtain
\[
\begin{bmatrix} q_B \\ f_B \end{bmatrix} = \Phi a = \Phi \text{diag}\left( \frac{1}{\mu_y - \lambda_y^i} \right) \Psi \begin{bmatrix} D_{BT}^{-1} & 0 \\ -D_{TT} D_{BT}^{-1} & I \end{bmatrix} \begin{bmatrix} e_B \\ e_T \end{bmatrix}.
\]  \tag{31}

Thus, the DOFs and nodal forces at \((\alpha, y)\) can be obtained, by utilizing the periodicity conditions imposed above, as
\[
\begin{bmatrix} q(\alpha, y) \\ f(\alpha, y) \end{bmatrix} = \Phi a(\alpha, y) = \Phi \text{diag}\left( \frac{e^{-ir_{\alpha}^s} e^{-ir_{\gamma}^s}}{\mu_y - \lambda_y^i} \right) \Psi \begin{bmatrix} D_{BT}^{-1} & 0 \\ -D_{TT} D_{BT}^{-1} & I \end{bmatrix} \begin{bmatrix} e_B \\ e_T \end{bmatrix}.
\]  \tag{32}

The above formulation will prove useful when treating the response to general loading which will be represented as a linear combination of CHPs.

3.2 General excitation

Now, the loading \( p(\alpha, y, r) \) is written as a superposition of CHPs as
\[
p(\alpha, y, r) = \frac{h(r)}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(\gamma_{\alpha}, \gamma_y) e^{-ir_{\alpha}^s} e^{-ir_{\gamma}^s} d\gamma_y,
\]  \tag{33}

where \( \gamma_{\alpha} = s \) is the circumferential wavenumber of the cylinder. For a cylinder, the phase change of a wave propagating around the circumference must be a multiple of \( 2\pi \). The circumferential wavemodes are independent and can be analysed individually. Furthermore, in Equation (33), \( \bar{g} \) is the two-dimensional Fourier transform of \( g(\alpha, y) \), given as
\[
\bar{g}(\gamma_{\alpha}, \gamma_y) = \frac{1}{2\pi} \int_{0}^{2\pi} \int_{-\infty}^{\infty} g(\alpha, y) e^{ir_{\alpha}^s} e^{ir_{\gamma}^s} d\alpha d\gamma_y
\]  \tag{34}

is the two-dimensional Fourier transform of \( g(\alpha, y) \). Consequently, the response to \( p(\alpha, y, r) \) is
\[ q(\alpha, y) = 1 \int_{-\infty}^{\infty} \sum_{y_{sN}}^N \Phi \text{diag} \left( \overline{g}(\gamma_{s+1}, \gamma_s) e^{-i\gamma_0 a_{s+1}} e^{-i\gamma_s y} \right) \Psi \begin{bmatrix} D_{BT}^{-1} & 0 \\ -D_{BT} D_{BT}^{-1} & \mathbf{I} \end{bmatrix} \begin{bmatrix} e_h \\ e_r \end{bmatrix} d\gamma_s, \] (35)

and \( e_h \) and \( e_r \) are obtained as described in subsection 3.1.

The integral in Equation (35) can be evaluated analytically using contour integration and the residue theorem. Moreover, for a closed cylindrical shell, the phase change of a wave propagating around the circumference must be a multiple of 2\( \pi \). Thus, the circumferential wavenumber can attain only integral values \( \gamma = s \) with \( s = 0, 1, 2, \ldots \), which defines the circumferential order. The circumferential wavemodes are independent and can be analysed on an individual basis. Consequently, Equation (35) becomes

\[ \begin{bmatrix} q(\alpha, y) \\ f(\alpha, y) \end{bmatrix} = \frac{1}{2\pi} \sum_{s=0}^{N} \Phi_s (\alpha, y), \] (36)

where \( \Phi_s \) (and \( \Psi_s \) in Equation (37) below) are the wavemodes of the cylinder at the \( s \)-th circumferential order. The infinite summation of Equation (35) is truncated at \( \gamma = s = \pm N \) where \( N \) is a high enough circumferential order such that, at frequency \( \omega \), the waves travelling in the \( y \)-direction are highly evanescent and thus have a negligible contribution to the response. The corresponding wave amplitudes are

\[ a_s(\alpha, y) = \int_{-\infty}^{\infty} \text{diag} \left( \overline{g}(\gamma_{s+1}, \gamma_s) e^{-i\gamma_0 a_{s+1}} e^{-i\gamma_s y} \right) \Psi_s \begin{bmatrix} D_{BT}^{-1} & 0 \\ -D_{BT} D_{BT}^{-1} & \mathbf{I} \end{bmatrix} \begin{bmatrix} e_h \\ e_r \end{bmatrix} d\gamma_s. \] (37)

The integral in Equation (37) can be evaluated using contour integration, and the result is

\[ a_{s,j}(\alpha, y) = \pm \frac{2\pi \overline{g}(\gamma_s, k_{y,j}^{s,j}) e^{-i\gamma_0 a_{s+1}} e^{-i\gamma_s y}}{\lambda_{s,j}^{s,j} \Delta y} \Psi_{s,j} \begin{bmatrix} D_{BT}^{-1} & 0 \\ -D_{BT} D_{BT}^{-1} & \mathbf{I} \end{bmatrix} \begin{bmatrix} e_h(\gamma_s, k_{y,j}^{s,j}) \\ e_r(\gamma_s, k_{y,j}^{s,j}) \end{bmatrix}, \] (38)

and \( j = 1, \ldots, m \). The above integration can be conducted using contour integration and the residue theorem as discussed in [21, 26].

Equation (19) can then be used to find the DOFs and internal forces. The above approach applies even if \( p(\alpha, y, r) = g(\alpha, y) h(r) \) is given numerically. The Fourier transform can be used to evaluate \( \overline{g}(\gamma_s, k_{y,j}^{s,j}) \) numerically. Moreover, the nodal forces \( e_h(\gamma_s, k_{y,j}^{s,j}) \) and \( e_r(\gamma_s, k_{y,j}^{s,j}) \) due to the external load can be evaluated numerically as well using Equation (23).

4. RESPONSE OF A FINITE STRUCTURE

The analysis for each given circumferential order in the WFE method presented above effectively one-dimensionalises the problem and allows previously developed techniques to be used to obtain the response of a finite right cylinder. Figure 3 shows the \((yz)\) plane view of a finite cylinder excited over a span \( L_0 \) by a distributed load (which can be applied in any direction). The wave amplitudes at various locations are also indicated. At each excitation frequency \( \omega \), and for the \( s \)-th circumferential order, the load generates positive- and negative-going waves of amplitudes \( a_s^+ \) and \( a_s^- \), respectively; these are obtained using Equation (38).
Figure 3 Wave amplitudes in a finite cylinder excited by an arbitrarily distributed load (hatched region).

Wave amplitudes $b_s^+$, for example, are the superposition of the directly excited waves of amplitudes $a_s^+$ and waves of amplitudes $g_s^+$ after propagating across the excited region. The boundary reflection matrices relate the incident and the reflected waves as $c_r^t = r_{s,t} c_s^t$ and $d_s^+ = r_{s,B} d_s^+$. Since any rotationally invariant boundary condition can be written as $Af + Bq = 0$ and the DOFs and forces can be projected onto the wave domain, the reflection matrices can be expressed as

\[
\begin{aligned}
    r_{s,t} &= -\left( A_{s,t} \Phi_{s,f} + B_{s,t} \Phi_{s,q} \right)^{-1} \left( A_{s,t} \Phi_{s,f} + B_{s,t} \Phi_{s,q} \right), \\
    r_{s,B} &= -\left( A_{s,B} \Phi_{s,f} + B_{s,B} \Phi_{s,q} \right)^{-1} \left( A_{s,B} \Phi_{s,f} + B_{s,B} \Phi_{s,q} \right).
\end{aligned}
\tag{39}
\]

The reflection matrices are frequency and circumferential order dependent.

The case of a distributed load will be treated first, Figure 3. The amplitudes of the directly excited positive- and negative-going waves generated by the distributed load acting over $L_y = y_2 - y_1$, $a_s^+(\alpha_r, y_2)$ and $a_s^-(\alpha_r, y_1)$ can be obtained via Equation (38), respectively. Then, the amplitudes of the incident waves $b_s$ and $g_s$ are given by

\[
\begin{aligned}
    b_s^+ &= \left[ I - \tau(y_2) r_{s,B} \tau(L) r_{s,t} \tau(L - y_2) \right]^{-1} \left[ a_s^+(\alpha_r, y_2) + \tau(y_2) r_{s,B} \tau(y_1) a_s^-(\alpha_r, y_1) \right], \\
    b_s^- &= \tau(L - y_2) r_{s,t} \tau(L - y_2) b_s^+ , \quad g_s^- = a_s^-(\alpha_r, y_1) + \tau(L_y) b_s^-, \quad g_s^+ = \tau(y_1) r_{s,B} \tau(y_1) g_s^-,
\end{aligned}
\tag{40}
\]

where $\tau(y) = \text{diag}\left(\exp(-i k_y^{-1} y), \ldots, \exp(-i k_y^{m} y)\right)$ is the wave propagation matrix that gives the amplitudes of the waves after propagating a distance $y$. The magnitudes of all the elements of $\tau(y)$ are $\leq 1$ with the elements corresponding to high order waves being nearly zero, thus ensuring good conditioning [16].

The response at $(\alpha_r, y_r)$ will depend on its location. If $y_r \geq y_2$, then

\[
\begin{aligned}
    h_s^+ &= \tau(y_r - y_2) b_s^+ , \quad h_s^- = \tau(L - y_r) r_{s,t} \tau(L - y_r) h_s^+ ,
\end{aligned}
\tag{41}
\]

are used to obtain the response at $y_r$. If $y_r \leq y_1$, the wave amplitudes

\[
\begin{aligned}
    h_s^- &= \tau(y_1 - y_r) g_s^- , \quad h_s^+ = \tau(y_r) r_{s,B} \tau(y_r) h_s^- ,
\end{aligned}
\tag{42}
\]

are used to evaluate the displacement. In all cases, the response at $(\alpha_r, y_r)$ is obtained as
\[
\begin{align*}
\{q(\alpha_r, y_r)\} &= \frac{1}{2\pi} \sum_{i=1}^{N} \Phi \{h_i^r\}, \\
\{f(\alpha_r, y_r)\} &= \\Phi \{h_i^s\}.
\end{align*}
\] (43)

For the case where \(y_1 < y_r < y_2\), the load can always be divided into two parts at \(y_r\). Then, the amplitude of the forced waves and incident waves at \(y_r\) can be obtained for each part of the load as per Equations (40) through Equation (43). Finally, the superposition principle is used to find the amplitude of the incident waves at \(y_r\). This is proposed in order to avoid inverting \(\tau(y_r)\) which might be singular as the elements corresponding to high order waves are nearly zero.

When a point force \(f_e\) (applied in a certain direction) is applied at \(y_r\), the amplitudes of the forced waves \(a_r^i(\alpha_r, y_r)\) and \(a_r^i(\alpha_r, y_r)\) can be obtained via Equation (38). Equations (40) through (42) still apply with \(y_1\) and \(y_2\) simply replaced by \(y_r\).

5. NUMERICAL EXAMPLES

In this section, three examples are presented to demonstrate the presented approach. The response of the first example, a simply supported (SS) isotropic cylinder, is available analytically. The second example is of an anti-symmetric cross-ply sandwich cylinder.

5.1 Isotropic cylinder

Now, the response of an isotropic cylinder is considered. The cylinder is made of mild steel with modulus of elasticity \(E = 200\) GPa, density \(\rho = 7800\) kg/m\(^3\), Poisson’s ratio \(\nu = 0.3\), loss factor \(\eta = 0.03\), thickness \(h = 1.8\) mm, mean radius \(R = 10\) cm, and length \(L = 2\) m. The mobility of the SS cylinder can be obtained using modal decomposition \([27]\). The resulting response is approximate since the displacements do not exactly satisfy the stress and displacement at the boundaries \([28]\). Thus, the WFE method will provide more accurate results. A non-dimensional frequency \(\Omega\) is defined as

\[
\Omega = \omega/\omega_r, \quad \text{where} \quad \omega_r = 1/R \sqrt{E/\rho(1-\nu^2)}
\] (44)

is the ring frequency.

For the WFE modelling, a single SHELL181 element of ANSYS\(^\circ\) is meshed with \(\Delta x = \Delta y = 1.75\) mm (i.e., \(\Delta \alpha = \Delta y/R = 5^\circ\)). This element has 4 nodes with 6 DOFs at each node. Thus, 6 pairs of waves are obtained when the eigenproblem of Section 2 is solved.

First, the response of the cylinder to a CHP is examined, Figure 4. Equations (31) and (32) indicate that the amplitude of the response is very high when the forcing wavenumber \(\gamma_y\) is close to the natural wavenumber of the structure \(k_y\). Figure 4 confirms this behaviour at increasing circumferential orders. At each frequency and as the circumferential order increases, the (absolute value of the) natural wavenumbers increase, and thus the peaks are further apart. The response is high near the natural wavenumbers and is very small away from these wavenumbers.

Now, a point force (acting in the radial direction) is applied at \((\alpha_r, y_r) = (0, L/2)\) and the radial response is observed at the input point and at \((\alpha_r, y_r) = (\pi/2, L/2)\). The input and transfer mobilities of the cylinder are shown in Figure 5. For the WFE model, only waves with \(\text{Im}(\tilde{k_y} \Delta y) \leq 1\) are retained for the reduced wave basis. For the modal solution, about
4000 modes were used for convergence. The modal density increases around the ring frequency, $\Omega = 1$. At higher frequencies, i.e., when $\Omega > 1$, the effect of curvature decreases and the response asymptotes to that of an infinite, flat plate.

$$\mathcal{W}$$

Figure 4 Response to unit amplitude CHP acting on the isotropic cylinder in the radial direction at (a) $\Omega = 0.1$ and (b) $\Omega = 1.5$. A log scale is used for the z-axis in both plots.

5.2 Antisymmetric cross-ply sandwich cylinder

This cylinder comprises a light, soft foam core sandwiched between two skins. The mean radius of the cylinder is $R = 1$ m and its length is $L_y = 1$ m. Each skin is made of four layers of graphite epoxy whose material properties were given in the previous example. The layup of the inner (i.e., bottom) skin is [0/90/90/0] degrees and that of the outer (i.e., top) skin is [90/0/0/90] degrees, and each skin is 4 mm thick. The core is a 10 mm polymethacrylamide ROHACELL foam which is isotropic with modulus of elasticity $E = 0.18$ GPa, density $\rho = 110$ kg/m$^3$ and Poisson’s ratio $\nu = 0.286$.

There have been attempts to analyse the wave behaviour of similar structures using the classical theory of sandwich structures [29] and using the Flügge theory [30] to describe the strain-displacement relations. This results in dispersions equations of the 47th and 42nd orders respectively which have to be subsequently solved numerically. For the WFE modelling, 18 SOLID45 elements of ANSYS® were meshed through the thickness of the cylinder with four elements for each skin and 10 elements for the core, with $\Delta x = \Delta y = 1$ mm. At each circumferential order, there are 57 pairs of admissible waves in the $y$–direction. Figure 6
shows a sample of the dispersion curves. The ring frequency of this cylinder is around 1150 Hz, and the waves cannot be simply categorised into axial, torsional or flexural wavemodes.

The cylinder is fixed at \( y = 0 \) and is free at \( y = L_y \). A point force (acting in the radial direction) is applied at \((\alpha_y, y_L) = (0, L_y)\) and the response of the cylinder is calculated. Figure 7 shows the input mobility of the cylinder in the radial direction. Analytical results for this model are extremely difficult to develop, and a full FE model would probably be very large. Indeed, as the length of the cylinder increases, the size of the FE model will increase proportionally. This makes the present approach particularly appealing when considering the vibrational behaviour of long pipes, submarines and aerospace structures. As for the example considered here, the input response of the cylinder is computed using the WFE approach and the predictions are compared to that of an infinite plate (with the same layup and properties), and to the response of the infinite cylinder. Above the ring frequency, the input response of the finite cylinder converges to that of the infinite cylinder which also converges to that of the plate as obtained using the approach presented in [21].

![Dispersion curves of the sandwich cylindrical shell at various circumferential orders](image)

**Figure 6** Dispersion curves of the sandwich cylindrical shell at various circumferential orders (a) \( s = 0 \) and (b) \( s = 1 \): ‘x’ real wavenumbers, ‘o’ imaginary wavenumbers.

![Input mobility of](image)

**Figure 7** Input mobility of: -- sandwiched cylinder, - - infinite sandwiched cylinder, ⋯ infinite sandwiched plate.
6. CONCLUSIONS

In this paper, a method for calculating the response of cylinders to arbitrary loads is presented. The method is based on the wave and finite element (WFE) method where a small segment of the cylinder is approximated as a flat rectangular segment which can be meshed using any number of shell or solid elements. The axisymmetry of cylinders is exploited and the finite element (FE) model of the small segment is post-processed using periodic structure theory to formulate an eigenproblem whose solution yields the wave properties of waves travelling along the axis of the cylinder at each circumferential order. The response of the cylinder is formulated as a linear combination of the response of the cylinder to a convected harmonic pressure (CHP). The latter is straightforward to obtain due to the homogeneity of the cylinder. Consequently, finding the response reduces to an inverse Fourier transform. Fortunately, the setup of the present WFE approach allows the a priori analysis of the wave behaviour of the cylinder at each circumferential order. This converts one of the integrals into a simple summation, and the remaining integral of the inverse Fourier transform can be evaluated analytically using contour integration and the residue theorem. Numerical examples were presented to demonstrate the approach which is very useful when considering cylinders with complicated constructions. The size of the treated WFE models is very small, especially if compared to full FE models at higher frequencies.

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