EVALUATION OF STATISTICAL OVERLAP AND FREQUENCY SPACING OF TWO RANDOMISED DYNAMIC SYSTEMS

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ABSTRACT
This work examines the statistical overlap and frequency spacing of structures with uncertainties. Two dynamic systems are examined corresponding to a simply-supported plate in flexural motion and a multi-degree of freedom system comprised of mass-spring oscillators coupled in series. For the plate, structural uncertainty was generated by mass perturbations. The mass-spring chain was randomised in two ways. In the first case, every individual mass was modified using a log-normal distribution. In the second case, small mass-spring systems were added at random locations to the main masses. The natural frequencies and frequency statistics are derived for each system. The probability distributions of the modal spacings are presented.

1. INTRODUCTION
Prediction of the dynamic responses of structures is required for transportation industries, with automotive and aerospace structures forming prime examples. There also exists a need to predict and attenuate the dynamic responses and radiated noise of white goods (fridges, washing machines, dishwashers) which affect the quality of the acoustical environment of the user. These structures are complex and structural uncertainties arise due to variability in material properties or from the manufacturing and assembly process. The existence of uncertainties in the structure properties can influence its dynamic responses (natural frequencies and frequency response functions). The difficulty in attempting to predict the dynamic responses of engineering structures is attributed to the large amount of physical variables which might be uncertain and the lack of data regarding the statistical distribution of these variables. Furthermore, for an ensemble of nominally identical structures such as vehicles off the production line, great variation in the vibrational response of each ensemble member occurs due to variability in the manufacturing and assembly processes. For example, the noise level inside a vehicle is closely related to the vibrations of the body shell. It has been found that successive vehicles from a production line can have noise levels that differ by 10 to 20 dB
across the low-to-mid frequency ranges [1]. The cause of the uncertainty is attributed in part to metal forming and spot welding techniques from the manufacturing process as well as variability in the properties of the soundproofing trim fitted to the vehicle [2].

There are a number of different approaches that have been recently presented to predict the response of systems with stochastic properties [3–5]. Typically, the first step to analyze the dynamic responses of a structure with uncertainties is to stochastically describe its natural frequencies and modeshapes. The models of uncertainty are generally based either on a parametric or non-parametric description of uncertainty, or combination of both. A parametric description of uncertainty means that the parameters of the dynamic system are taken to be uncertain variables that can be statistically described using, for example, perturbation methods [6-8], interval analyses [9-11], or fuzzy structure theory [12-14]. Uncertainty is then propagated through the equations of motions using various techniques, which usually involves high computational cost. On the other hand, non-parametric analysis of uncertainty assumes that regardless of their detailed nature, the uncertainties in the system can be described with some kind of “universal uncertainty” model. A wide range of literature can be found on such non-parametric methods [15-22].

Early research into non-parametric models for dynamic structures with uncertainties describes the distribution of the spacing between successive natural frequencies by a Poisson point process model [15, 16]. Using the Poisson point process, the probability density function (pdf) of the spacing between successive natural frequencies is described by an exponential distribution. However, the Poisson model is only valid for a system with symmetries, such as a perfectly rectangular plate or a perfectly box-shaped room [17]. In such symmetric systems, frequencies from different symmetry groups have a high possibility to coincide, thus resulting in small modal spacings. It was later discovered within random matrix theory that the Gaussian orthogonal ensemble (GOE) is more realistic to describe the modal statistics of engineering structures. The GOE is an ensemble of random symmetric matrices whose entries are Gaussian, zero mean and statistically independent. The off-diagonal entries have a common variance and the diagonal entries have twice this variance [2, 17]. A feature of the GOE is that the spacing between successive natural frequencies is described by a Rayleigh distribution. There is a large amount of literature investigating the applicability of the GOE to dynamic systems. A pioneering paper was presented by Weaver [18], in which he investigated aluminium blocks with slits cut into them and showed a transition in the modal spacing statistics from an exponential distribution to a Rayleigh distribution with increasing structural irregularity. The GOE was found applicable to a range of non-symmetric structures [19-22].

This paper examines the natural frequency statistics of a plate and a mass-spring chain, which are randomised by mass or mass-spring perturbations. Small masses were added at random locations to the plate. For the mass-spring chain, every individual mass was modified using a log-normal distribution. A further case for the mass-spring chain was examined in which a small mass-spring system was added at random locations to the main masses. For each dynamic system, uncertainty across an ensemble was generated in order to observe the statistical overlap factor of the natural frequencies and the modal spacing statistics.

2. EXPONENTIAL AND RAYLEIGH DISTRIBUTIONS

The modal spacing statistics of a physically symmetric dynamic or acoustic system such as a perfectly rectangular vibrating plate or a perfectly box-shaped room results in the probability density function (pdf) of the spacings between successive natural frequencies described by an
exponential distribution, which is given by [15]

\[ p(s) = ae^{-as}, \ s \geq 0 \]  

(1)

where \( a = 1/\mu \), \( s \geq 0 \) and \( \mu \) is the mean spacing between adjacent natural frequencies. In a Rayleigh distribution, the pdf of the spacings between successive natural frequencies is given by [17]

\[ p(s) = \frac{s}{c^2} e^{-s^2/2c^2}, \ s \geq 0 \]  

(2)

where \( c = \mu \sqrt{2/\pi} \).

3. DYNAMIC SYSTEMS

3.1 Plate with added masses

Using the derivation by Meirovitch [23] and Brown [24], the Lagrange-Rayleigh-Ritz technique is used to derive the equation of motion in modal space for a plate with randomly located point masses, as shown in Fig. 1.

![Figure 1. A simply supported plate with randomly located point masses](image)

The flexural displacement of a bare rectangular plate in modal space is given by

\[ w(x, y, t) = \sum_{mn} q_{mn}(t)\psi_{mn}(x) \]  

(3)

where \( q_{mn} \) is the modal coordinate. \( \psi_{mn}(x) = \psi_m(x)\psi_n(y) \) are the mass-normalized mode shapes which satisfy the orthogonal condition

\[ \int_0^{L_x} \int_0^{L_y} \rho h \psi_{mn} \psi_{m'n'} dxdy = \begin{cases} 1, & mn = m'n' \\ 0, & mn \neq m'n' \end{cases} \]  

(4)

\( \rho \) is the plate density, \( h \) is the thickness and \( L_x, L_y \) are the lengths in the \( x \) and \( y \) directions, respectively. For a simply supported plate, the mass-normalized eigenfunctions are given by

\[ \psi_{mn}(x) = \frac{1}{\sqrt{M_n}} \phi_{mn}(x) = \frac{1}{\sqrt{M_n}} \sin \left( \frac{mx}{L_x} \right) \sin \left( \frac{ny}{L_y} \right). \]  

(5)
where $M_n = \rho h L_x L_y / 4$ is the modal mass. $\omega_{mn} = \sqrt{D / \rho h \left( (m\pi/L_x)^2 + (n\pi/L_y)^2 \right)}$ are the natural frequencies of the bare simply supported plate, $D = \frac{E h^3}{12(1-\nu^2)}$ is the plate flexural rigidity, and $E$, $\nu$ are respectively Young’s modulus and Poisson’s ratio.

The kinetic energy of a plate with $N_m$ number of added point masses is given by

$$T = \frac{\rho h}{2} \int_0^{L_x} \int_0^{L_y} \dot{w}^2(x) \, dx \, dy + \sum_{a=1}^{N_m} m_a \dot{w}^2(x_a)$$

where $\dot{w}$ denotes the derivative of the plate flexural displacement $w$ with respect to time, $m_a$ is the size of each added mass and $x_a = (x_a, y_a)$ is the location of each added mass. Using the orthogonality condition the kinetic energy of the simply-supported plate can be expressed as

$$T = \frac{1}{2} \sum_{mn} \dot{q}_{mn}^2 + \frac{1}{2} \sum_{a=1}^{N_m} \sum_{mn} m_a \dot{q}_{mn} \dot{q}_{jk} \psi_{mn}(x_a) \psi_{jk}(x_a).$$

Similarly, the potential energy of a plate with $N_m$ number of point masses can be expressed as

$$V = \frac{1}{2} \sum_{mn} \omega_{mn}^2 q_{mn}^2.$$  

Lagrange’s equation for a particular modal coordinate $pq$ of the plate is given by [23]

$$\frac{\partial}{\partial t} \left( \frac{\partial T}{\partial \dot{q}_{pq}} \right) - \frac{\partial T}{\partial q_{pq}} + \frac{\partial V}{\partial q_{pq}} = 0$$

Differentiating the kinetic and potential energies given by Eqs. (8) and (9) with respect to the modal coordinate $pq$ and substituting into Lagrange’s equation results in the equation of motion of the plate with point masses [25].

$$\ddot{q}_{pq} + \sum_{a=1}^{N_m} \sum_{mn} m_a \dot{q}_{mn} \psi_{mn}(x_a) \psi_{pq}(x_a) + \omega_{pq}^2 q_{pq} = 0$$

### 3.2 Mass-spring chain

A finite mass-spring chain is defined by a one-dimensional array of masses $m_n$ connected by springs of stiffness $k_n$, as shown in Fig. 2.

![Figure 2. A finite mass-spring chain](image-url)
The equation of motion of the mass-spring chain is given by

$$(K_m - \omega_m^2 M_m)X = 0 \quad (11)$$

where $K_m$ and $M_m$ are the global stiffness and mass matrices given in what follows

$$M_m = \begin{bmatrix} m_1 & & & \\ & m_2 & & \vdots \\ & & \ddots & \vdots \\ & & & m_{n-1} \end{bmatrix} \quad (12)$$

$$K_m = \begin{bmatrix} k_1 + k_2 & -k_2 & & & \\ -k_2 & k_2 + k_3 & -k_3 & & \\ & \ddots & \ddots & \ddots & \\ & & k_{n-1} + k_n & -k_n & \\ 0 & & & k_n + k_{n+1} \end{bmatrix} \quad (13)$$

and $X$ is the displacement vector. The natural frequencies of the mass-spring chain were obtained in Matlab by solving $(K_m - \omega_m^2 M_m)X = 0$.

4. NUMERICAL RESULTS

4.1 Simulation parameters

For the mass-loaded plate, the properties of steel are used and are listed in Table 1. Damping is included using a complex Young’s modulus $E(1 + j\eta)$ where $\eta$ is the structural loss factor and $j = \sqrt{-1}$ is the imaginary unit. The plate is randomised by added point masses in two ways. In the first case, plates with $n = 10, 20, 50$ added masses at random locations are examined, where each mass is 1% of the bare plate mass. The location of the added masses corresponding to their coordinates on the $x$ and $y$ axes, are selected using a uniform distribution. In the second case, plates with $n = 10, 20, 50$ added masses at random locations are examined, where the total amount of added mass is constant and corresponds to 10% of the bare plate mass. For each case, 50 sets of data are generated from which the mean natural frequencies, their standard deviation and the mean frequency spacing across the ensemble are obtained.

The mass-spring chain consists of $n = 150$ nominally identical masses ($m_c = 5$kg), connected by 149 nominally identical springs ($k_c = 1 \times 10^7$N/m). At both ends, the chain is supported by a spring with stiffness $2k_c$. The structural loss factor is 0.01. The mass-spring chain is randomised in two ways. In the first case, every individual mass is modified using a log-normal distribution whose mean is the nominal value. The standard deviations, denoted by $\sigma_m$, are normalised by the mean and range from 0.1 to 0.99. In the second case, the locations of a specified number of main masses $m_c$ are randomly chosen based on a uniform distribution. A spring-mass system comprising a spring of stiffness $k_r$, structural loss factor
0.01 and mass $m_r$ is added to the randomly selected main masses. The randomly attached mass-spring systems increase the number of degrees of freedom of the system. Uncertainty in the locations of the added spring-mass systems across an ensemble of 50 mass-spring chains was generated from which the natural frequencies were obtained.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$L_x$</td>
<td>1.35 m</td>
</tr>
<tr>
<td>$L_y$</td>
<td>1.2 m</td>
</tr>
<tr>
<td>$h$</td>
<td>5 mm</td>
</tr>
<tr>
<td>$E$</td>
<td>210 GPa</td>
</tr>
<tr>
<td>$\rho$</td>
<td>7800 kg/m$^3$</td>
</tr>
<tr>
<td>$v$</td>
<td>0.3</td>
</tr>
<tr>
<td>$\eta$</td>
<td>0.1%</td>
</tr>
</tbody>
</table>

Table 1: Parameters of the mass-loaded plate

$$p(m) = \frac{1}{m\sqrt{2\pi}} e^{-\frac{(\ln m - \mu)^2}{2\varepsilon^2}}, \quad m \geq 0$$  \hspace{1cm} (14)$$

$$\text{mean}(m) = e^{(\mu + \varepsilon^2/2)}$$  \hspace{1cm} (15)$$

$$\text{var}(m) = e^{(2\mu + 2\varepsilon^2)} - e^{(2\mu + \varepsilon^2)}$$  \hspace{1cm} (16)$$

$$\sigma_m = \frac{\sqrt{\text{var}(m)}}{\text{mean}(m)}$$  \hspace{1cm} (17)$$

![Figure 3. A finite mass-spring chain with randomly connected masses and springs](image)

### 4.2 Statistical overlap factor

The statistical overlap factor is defined by [26]

$$S = \frac{2\sigma}{\mu} = \frac{2[\text{var}(\omega_n)]^{1/2}}{\text{mean}[\omega_n - \omega_{n-1}]}$$  \hspace{1cm} (18)$$

where in the stochastic system, $\sigma$ is the standard deviation of the natural frequencies $\omega_n$ from the mean value caused by structural uncertainties, and is measured across an ensemble of random structures. $\mu$ is the mean frequency spacing between adjacent natural frequencies.
The statistical overlap factor is a useful parameter to measure the amount of mixing and veering between modes across an ensemble of nominally identical systems with uncertainties.

### 4.2.1 Plate with added masses

In the modal analysis of the mass-loaded plate, 1225 modes of the plate were obtained from which the top 20% of modes were discarded as the modal spacing greatly increases for the highest order modes [23]. The statistical overlap factor analysis for the plate focused on the frequency range up to 10 kHz. For each case, 50 sets of data were generated from which the mean natural frequencies, their standard deviation and the mean frequency spacing across the ensemble were obtained to analyse statistical overlap factors.

The statistical overlap factors of the mass-loaded plate for the first case in which \( n = 10, 20 \) and 50 masses are added to the bare plate, and where every added mass is 1% of the bare plate mass, are shown in Fig. 4a. The statistical overlap factors of the mass-loaded plate for the second case, in which \( n = 10, 20 \) and 50 masses are added to the bare plate and the total added mass is equal to 10% of the bare plate mass, are shown in Fig. 4b.

As shown in the figures, the statistical overlap factor curve of the plate consists of two parts, the linear part and the saturated part. At lower frequencies, the modes of the plate are well defined, and as such are not greatly influenced by structural uncertainty. As the frequency increases, the structure becomes more sensitive to any uncertainty. The statistical overlap factor increases steeply and then levels off around unity. In each case, after a certain frequency, the statistical overlap factor becomes saturated and does not further increase with frequency. Observation of Fig. 4a shows that the slope of the statistical overlap factor slightly increases with increasing randomness, due to increasing the number of masses of the same size, thereby increasing the total added mass.

![Figure 4. Statistical overlap factors for the mass-loaded plate](image)

In Fig. 4b, the amount of total added mass is constant, which is achieved using a smaller number of larger sized added masses or a larger number of smaller sized masses. For the example shown here, the slope of the statistical overlap factor slightly increases using a smaller number of larger sized added masses. However, it could be expected that increasing the size of the added mass would effective clamp the plate at the locations of the added masses and reduce the structural uncertainty.
4.2.2 Mass-spring chain

The natural frequencies of the mass-spring chain were obtained directly from its mass and stiffness matrices. For the original mass-spring chain (in the absence of any added spring-mass systems), the \( n \)th modal frequency can be expressed by [3]:

\[
f_n = \frac{\sin\left(\frac{n\pi}{300}\right)}{\pi} \sqrt{\frac{k_c}{m_c}}
\]

(19)

where \( n = \{1,2,\cdots,150\} \). The maximum modal frequency of the original mass-spring chain (of \( m_c = 5\text{kg}, \ k_c = 1 \times 10^7\text{N/m} \)) is 450Hz.

The statistical overlap factors of the mass-spring chain for the first case where every individual mass is modified using a log-normal distribution, are shown in Fig. 5. When the mass deviations are small (Fig 5a), the statistical overlap factors of the mass-spring chain increase steadily with increasing frequency and larger variance of the log-normal distribution. As the variance increases (Fig. 5b), the statistical overlap factor remains almost unchanged and levels off around a value of 2.

![Figure 5. Statistical overlap factors for the mass-spring chain](image)

In the second case, different arrangements for the added spring-mass systems to the main mass-spring chain were examined corresponding to (i) varying the number of added spring-mass systems, (ii) varying the size of the added masses, (iii) varying the stiffness of the added springs, and (iv) varying the size of the added masses and springs but keeping the ratio of \( k_r \) and \( m_r \) constant. The various arrangements for the added spring-mass systems are listed in Table 2.

<table>
<thead>
<tr>
<th>Case</th>
<th>Number of attached mass-spring systems</th>
<th>Mass ( m_r ) (kg)</th>
<th>Stiffness ( k_r ) (( \times 10^7 ) N/m)</th>
</tr>
</thead>
<tbody>
<tr>
<td>a</td>
<td>10, 20, 50</td>
<td>0.5</td>
<td>0.1</td>
</tr>
<tr>
<td>b</td>
<td>20</td>
<td>0.25, 0.5, 1, 1.5</td>
<td>0.05, 0.1, 0.2, 0.3</td>
</tr>
<tr>
<td>c</td>
<td>20</td>
<td>0.2, 0.5, 1, 2</td>
<td>0.1</td>
</tr>
<tr>
<td>d</td>
<td>20</td>
<td>0.5</td>
<td>0.05, 0.1, 0.2, 0.3</td>
</tr>
</tbody>
</table>

Table 2: Parameters of the randomly attached mass-spring systems
The statistical overlap factors of the mass-spring chain for each arrangement in Table 2 can be observed in Fig. 6. The peaks in the statistical overlap factor curves correspond to the natural frequencies of the individual, randomly attached, mass-spring systems which is given by

\[ f_r = \frac{1}{2\pi} \sqrt{\frac{k_r}{m_r}} \]  

(20)

In Fig. 6a, \( f_r = 225.08 \text{ Hz} \) for \( m_r = 0.5 \text{ kg} \) and \( k_r = 0.1 \times 10^7 \text{ N/m} \). Increasing the number of added mass-spring systems tends to increase the amplitude of the peak. For a fixed ratio of \( k_r \) and \( m_r \), the amplitude of the peak in the statistical overlap factor remains fairly constant (Fig. 6b). As the size of the added masses is increased, the peak of the statistical overlap factor shifts to lower frequencies (Fig. 6c). Similarly, as the stiffness of the added springs increases, the peak shifts to higher frequencies (Fig. 6d).

![Figure 6. Statistical overlap factors for the mass-spring chain](image)

**4.3 Natural frequency spacings**

**4.3.1 Plate with added masses**

The probability density function (pdf) of the frequency spacings for one structure in each ensemble of the mass-loaded plates are shown in Fig. 7. A pdf of the frequency spacings for each structure was generated as follows: the natural frequencies obtained from eigenvalue
analysis were arranged in ascending order and the spacings between successive frequencies were obtained. The top 20% of modes were discarded and the mean frequency spacing was calculated to generate a Rayleigh or exponential distribution. A histogram of the natural frequency spacings was then generated and converted to a pdf by scaling to unit area.

Figure 7. pdf of frequency spacings and Rayleigh distribution for the mass-loaded plate
The modal spacing statistics of a bare simply supported rectangular plate is shown in Fig. 7a, and matches well with the exponential distribution of its mean frequency spacing. The pdfs for the mass-loaded plates are obtained in the range from 1 kHz to 10 kHz, where the statistical overlap factors for each ensemble remains fairly constant. Figures 7b-7f show that the pdfs of the frequency spacings for the plates with mass perturbations exhibit a good match with the Rayleigh distribution curve in terms of the mean frequency spacing. The values of the mean frequency spacings for each ensemble are listed in Table 3 and are reasonably constant.

<table>
<thead>
<tr>
<th>Added masses</th>
<th>Mean frequency spacing (Hz)</th>
</tr>
</thead>
<tbody>
<tr>
<td>10 masses (1%)</td>
<td>10.35</td>
</tr>
<tr>
<td>20 masses (1%)</td>
<td>10.32</td>
</tr>
<tr>
<td>50 masses (1%)</td>
<td>10.23</td>
</tr>
<tr>
<td>20 masses (0.5%)</td>
<td>10.32</td>
</tr>
<tr>
<td>50 masses (0.2%)</td>
<td>10.25</td>
</tr>
</tbody>
</table>

Table 3: Mean frequency spacings of the mass-loaded plates

### 4.3.2 Mass-spring chain

The mean value of the frequency spacing is directly influenced by the variance of the randomised masses, as shown in Table 4. The pdf of the frequency spacings results in a poor match with the Rayleigh distribution using the mean frequency spacings for each case. The results are not shown here. The poor match is attributed to the fact that the mass-spring chain has a limited number of degrees of freedom and its natural frequencies are well defined. Even though a log-normal distribution with large variance applied to the masses in the mass-spring chain results in the curves for the statistical overlap factor levelling off to a constant value, the frequencies are limited to the low frequency range to which the GOE doesn’t apply.

<table>
<thead>
<tr>
<th>Variance of log-normal distribution</th>
<th>Mean frequency spacing (Hz)</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \sigma_m = 0.8 )</td>
<td>4.90</td>
</tr>
<tr>
<td>( \sigma_m = 0.9 )</td>
<td>4.78</td>
</tr>
<tr>
<td>( \sigma_m = 0.95 )</td>
<td>5.64</td>
</tr>
<tr>
<td>( \sigma_m = 0.99 )</td>
<td>6.27</td>
</tr>
</tbody>
</table>

Table 4: Mean frequency spacings of the mass-spring chain

### 5. CONCLUSIONS

This paper investigates the natural frequency statistics of two dynamic systems with structural uncertainties corresponding to a mass-loaded plate and a mass-spring chain, which are randomised by mass or mass-spring perturbations. For the plate, the probability density function of the frequency spacings is altered from an exponential distribution to a Rayleigh distribution with the attachment of small masses at random locations. As the frequency increases, different modes for the mass-loaded plate strongly mix with and veer from each other, its statistical overlap factor becomes saturated, and its frequency spacing distribution is suitably predicted by the GOE model. For the mass-spring chain, the number of modes is
limited, they are in well defined positions and are limited to the low frequency range. As such, the modal statistics for the mass-spring chain are not suitably predicted by the GOE.

REFERENCES


