AEROELASTIC STABILITY
OF A SYMMETRIC MULTI-BODY SECTIONAL MODEL

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ABSTRACT
The aeroelastic stability of a multi-body system is studied through a four-degree-of-freedom model, which describes the linearized section dynamics of particular suspended bridges with doubly-symmetric cross-section, subject to a lateral stationary wind flow. A multi-parameter perturbation solution, applied on the classic modal problem in internal resonance conditions, allows a consistent reduction of the model dimensions. The bifurcation scenario described by a local stability analysis in the resonance region is featured by two stability boundaries, strongly interacting in the parameter space. The analysis of the critical wind velocity provides satisfying engineering results, pointing out how resonant light cables or dissipative cable-deck couplings may have beneficial effects in preventing the aeroelastic instability.

1. INTRODUCTION
Dynamic analysis of two or more interconnected rigid or flexible bodies, also known as multi-body systems, is a topic of great interest with applications in a broad variety of engineering fields. On the other hand, section models have often been applied in wind engineering in order to perform aerodynamic stability analyses using the quasi-steady theory. The classic phenomenon of aerodynamic instability involving slender structures with noncircular cross-sections is called galloping [1]; it is classified as a velocity-dependent, damping-controlled instability, giving rise to transverse or torsional motions [2],[3]. The applicability of the quasi-steady theory has been deeply discussed [1],[4], and its employment for torsional oscillations is often judged inadequate to provide realistic results, even if it can succeed in predicting the onset of torsional galloping [1]. Despite its inherent shortcomings, the common use of this approach can still be justified by the valuable possibility to assess the system aerodynamic stability through standard static experimental tests.

The present paper approaches the aeroelastic stability analysis of a multi-body four-degree-of-freedom model, which appears able to synthetically reproduce the linearized dynamics of particular suspended bridges with doubly-symmetric cross-section. The model simultaneously describes both the internal mechanical interactions among the deck section (principal body) and two pre-tensioned hanger cables (secondary bodies), caused by the geometric stiffness, and the aerodynamic coupling between the vertical flexion and torsion components of the deck motion, owing to a stationary wind flow acting along the horizontal symmetry axis.
Multi-parameter perturbation methods allow the asymptotic approximation of the eigen-properties governing the system undamped free dynamics. In particular regions of the parameter space, internal resonance conditions occur, involving global modes, dominated by the two components of the deck motion, and a pair of local modes, dominated by the cable transversal motion. Within this framework, the specific goal of the present work is to analyze the unexplored effects of the cable-deck interactions on the aeroelastic stability of the entire system. To this purposes, the critical conditions are imposed on the resonant system, reduced to modal coordinates, to perform a local stability analysis leading to a stability chart in the extended parameter space including the wind mean velocity and all the damping terms. In particular, the influence of a slight detuning between the natural frequencies (nearly-resonant systems) on the critical wind velocity is investigated and the effects of (small) mass and (low) damping properties of the cables on the instability boundaries are discussed.

2. MULTI-BODY SECTIONAL MODEL

The cable-deck structural interactions in a suspended or cable-stayed bridge typically involve the three-dimensional deck motion and the transversal motion of one or more resonant cables. A dynamic multi-body model composed by a principal system (SP), represented by a doubly-symmetric rigid rectangular body, and a pair of secondary systems (SS\(_1\) and SS\(_2\)), represented by identical point bodies, can be effectively employed to synthetically reproduce the complex scenario of linear and nonlinear coupling among the rigid planar sections of the bridge deck and the flexible cables connecting them to the supporting system, namely the main cables (suspended bridges), the towers (cable-stayed bridges), or the superior arch (arch bridges) [6].

The principal system is supposed rigidly constrained in the horizontal direction. Therefore its dynamic configuration is fully defined by the centroid vertical displacement \(V\) and rotation \(\theta\) (Figure 1b). The elastic constant \(K_{pi}\) of the two linear springs (i=1,2) connecting the principal system to the lower ground simulate the flexural and torsional stiffness of the bridge deck. The height-to-width ratio of the rectangular shape and the material mass density can be independently varied to properly capture the vertical \(M_p\) and rotational inertia \(J_p\) of the deck section. The dynamic configuration of each secondary system is described by its vertical \(V_i\) and horizontal displacement \(U_i\) (Figure 1b). Two springs connect each secondary system, with mass \(M_s\), to the principal system and the upper ground, simulating the anchorages to the deck and the (quite rigid) suspension system, respectively. The spring elastic constant \(K_{sij}\) (i,j=1,2) simulates the cable axial stiffness, whereas the geometric stiffness acting on the transversal cable motion can be accounted for by introducing a spring prestress \(H_{sij}\) (i,j=1,2).

![Figure 1. Multi-body system: (a) static and (b) dynamic configuration, (c) aerodynamic forces](image)
Moving from an exact formulation of the model finite kinematics, the nonlinear equations governing the system dynamics are obtained. These equations can be linearized around the initial prestressed configuration, whose static self-equilibrium, in absence of dead external forces, requires the two conditions \( H_1=H_{p1}=H_{s11}=H_{s12} \) and \( H_2=H_{p2}=H_{s21}=H_{s22} \). The nondimensional dependent and independent variables can be introduced

\[
\tau = \omega_p t, \quad v = \frac{V}{L_s}, \quad u_1 = \frac{U_1}{L_s}, \quad u_2 = \frac{U_2}{L_s}, \quad v_1 = \frac{V_1}{L_s}, \quad v_2 = \frac{V_2}{L_s}, \quad \text{where} \quad \omega_p^2 = \frac{K_t}{M_p} \]

(1)

and the following nondimensional elastic and geometric parameters can be defined

\[
\alpha = \frac{A}{B}, \quad \beta = \frac{B}{L_s}, \quad \delta = \frac{L_p}{L_s}, \quad \rho^2 = \frac{M_s}{M_p}, \quad \chi^2 = \frac{J}{M_p L_s^2}, \quad \mu_{1,2}^2 = \frac{H_{1,2}}{K_t L_s}, \quad \omega_\nu^2 = \frac{K_t + K_s}{K_t}, \quad \kappa = \frac{K_t}{K_t}
\]

(2)

where the hypothesis \( K_s=K_{s11}=K_{s12}=K_{s21}=K_{s22} \) has been introduced for sake of simplicity, and the auxiliary parameters \( K_t=K_{p1}+K_{p2}, \quad K_s=K_{p2}-K_{p1} \) have been defined. Finally, accepting the reasonable assumption of small geometric-to-elastic stiffness ratio in the cables \( (H_{1,2} \ll K_t L_s) \), the vertical motion of the secondary systems can be statically condensed in the low-frequency range of the system dynamics, in order to obtain a four-degree-of-freedom (4-dofs) model, in which the displacement vector is \( \mathbf{u}=(v, \theta, u_1, u_2)^T \). Under the above considerations, the linear equation of motion governing the system forced response reads

\[
\mathbf{M} \ddot{\mathbf{u}} + \mathbf{C} \dot{\mathbf{u}} + \mathbf{K} \mathbf{u} = \mathbf{f}(\mathbf{u}, \dot{\mathbf{u}})
\]

(3)

where the non-null coefficients of the mass matrix \( \mathbf{M} \) and the stiffness matrix \( \mathbf{K} \) are reported in the Appendix. The viscous damping matrix \( \mathbf{C} \) describes the structural damping, as well as the additional dissipative effect which may be introduced through damper connections among two or more dofs. The aerodynamic forces \( \mathbf{f}(\mathbf{u}, \dot{\mathbf{u}}) \) are defined in the following paragraph.

### 2.1 Aerodynamic forces

The stationary flow acting on the doubly-symmetric section can be physically characterized by the air density \( d_A \) and the mean wind velocity \( U_A \). Denoting with \( c_d \) the sectional drag coefficient, \( c'_l \) and \( c'_m \) the first derivatives of the section lift and torsional moment coefficients (with respect to the angle of attack \( \theta_A \) of the relative velocity \( V_A \), see Figure 1c), and defining \( B_A \) a suitable reference length of the cross-section and \( R_A \) its characteristic radius [4], the aerodynamic forces assume the non-dimensional simplified expressions (e.g. [2],[5])

\[
\begin{align*}
    f_A &= -\frac{1}{2} k_A u_A [u_A c'_l \dot{\theta} + (c_d + c'_l) \dot{v} - r_A (c_d + c'_l) \dot{\theta}] \\
    m_A &= -\frac{1}{2} k_A u_A \beta_A c'_m [u_A \theta + \dot{v} - r_A \dot{\theta}], \quad \text{where} \quad k_A = \frac{1}{4} \beta \frac{\rho A}{\alpha \beta} \omega_p^2 \frac{B_A}{L_s}, \quad u_A = \frac{U_A}{\omega_p L_s}, \quad r_A = \frac{R_A}{L_s}, \quad \rho_A = \frac{d_A AB}{M_p}
\end{align*}
\]

(4)

where the aerodynamic dimensionless quantities are defined as:

\[
\beta_A = \frac{B_A}{L_s}, \quad u_A = \frac{U_A}{\omega_p L_s}, \quad r_A = \frac{R_A}{L_s}, \quad \rho_A = \frac{4 d_A AB}{M_p}
\]

(5)

Therefore, casting the displacement and velocity coefficients into the aerodynamic stiffness and damping matrices \( \mathbf{K}_A \) and \( \mathbf{C}_A \), respectively, the non-dimensional linearized equations of motion (3) can be reformulated as:

\[
\mathbf{M} \ddot{\mathbf{u}} + (\mathbf{C}_A+\mathbf{C}_A) \dot{\mathbf{u}} + (\mathbf{K}_A+\mathbf{K}_A) \mathbf{u} = \mathbf{0}
\]

(6)

where the non-zero terms of the matrices \( \mathbf{C}_A \) and \( \mathbf{K}_A \) are defined in the Appendix.
3. CLASSICAL MODAL ANALYSIS

Neglecting the aerodynamic forces and the structural damping, a classic modal analysis can be performed. Decomposing the mass matrix in the form $\mathbf{M} = \mathbf{Q}^\top \mathbf{Q}$ (the decomposition is unique as the matrix $\mathbf{M}$ is diagonal), the real eigenvalues $\lambda$ and eigenvectors $\mathbf{\psi}$ characterizing the free undamped vibrations ensue from the solution of the standard eigenproblem

$$ \left( \mathbf{G} - \lambda \mathbf{I} \right) \mathbf{\varphi} = 0 $$

where $\mathbf{\varphi} = \mathbf{Q} \mathbf{\psi}$, and the equivalent stiffness matrix is $\mathbf{G} = \mathbf{Q}^\top \mathbf{K} \mathbf{Q}^{-1}$. The exact eigensolution loci can be traced by solving the eigenproblem (7) throughout the whole technically relevant parameter range, for instance recurring to numerical continuation techniques.

Parametric analyses of the solution show that a generic parameter set typically corresponds to well-distinct natural frequencies, related to modes dominated either by one of the principal system components of motion (global modes, Figure 2a), or by the horizontal displacement of one or both the secondary systems (local modes, Figure 2b,c) [6]. Particular regions of the parameter space are instead associated to internal resonance or nearly-resonance conditions, which correspond to identical or close frequencies. Within these resonant regions the system eigensolution exhibits rapid modifications under small changes of the varying parameters, that is, high eigensolution sensitivity. Therefore, particular care must be paid to the application of perturbation methods, which are nonetheless an attractive and powerful alternative to perform local parametric analysis within the resonant regions of the parameter space.

3.1 Perturbation solution

A perturbation approach to the solution of the eigenproblem requires a suited ordering of the significant parameters, through the introduction of a small scaling parameter $\epsilon \ll 1$

$$ \beta = \epsilon \tilde{\beta}, \quad \rho^2 = \epsilon^2 \, \rho_0^2 (1 + \epsilon \sigma), \quad \chi^2 = \epsilon^2 \chi^2, \quad \mu_1^2 = \epsilon^2 \mu_1^2, \quad \mu_2^2 = \epsilon^2 \mu_2^2 (1 + \epsilon \eta), \quad \kappa = \epsilon \kappa $$

From a physical viewpoint, this parameter ordering in the multi-body system can be justified by engineering reasons, as it accounts for, first, the lightness and transversal flexibility of the two cables, which differ to each other for a slight pretension difference only (described by the new parameter $\eta$), and, second, the flatness of the typical bridge deck sections, which are also characterized by small torsional inertia and stiffness. After the introduction of the parameter ordering and the expansion in $\epsilon$-powers, the equivalent stiffness matrix can be expressed as

$$ \mathbf{G} = \mathbf{G}_0 + \epsilon \, \mathbf{G}_1 + \mathcal{O}(\epsilon^2) $$

where the non null coefficients of the unperturbed stiffness $\mathbf{G}_0$ and the stiffness perturbation $\mathbf{G}_1$ are reported in the Appendix. It is worth noting that $\mathbf{G}_0$ is a diagonal matrix. Therefore the unperturbed system is composed of independent oscillators, whose natural frequencies can be tuned to each other by equating the diagonal terms, in order to obtain exact internal resonance
conditions among one or more system dofs. The frequencies of the two secondary systems, in particular, are always resonant, since they are described by identical oscillators, due to the smallness of the pretension difference in the springs. Distinguishing the frequencies of the vertical displacement (V) and rotation (T) of the principal system from those of the horizontal displacements (UU) of the secondary systems, the internal resonance conditions read

\[
VT: \frac{\alpha \beta}{\chi} = 1, \quad VUU: \frac{\alpha^2 \rho_0^2}{2 \mu^2} = 1, \quad TUU: \frac{\alpha \beta}{2 \mu^2} = 1, \quad VTUU: \frac{\alpha \beta}{\chi} = 1
\]

where the tilde has been omitted. These conditions define the one or two-dimensional loci of the parameter space corresponding to exactly resonant unperturbed systems. When the small stiffness perturbation \( G_1 \) is added, the multi-body model moves within the (narrow) parameter region which surrounds these loci and corresponds instead to nearly-resonant systems.

To the specific purposes of this work, focus is made on the TUU resonance region, by imposing the following relations among the parameters, equivalent to the condition (10)c

\[
\omega^2 \left( \frac{2 \mu^2}{\rho_0^2} \right) = \omega^2, \quad \Gamma = \left( \frac{\chi}{\alpha \beta} \right) \neq 1
\]

where \( \omega \) is the triple frequency of the unperturbed system, and \( \Gamma \) has to be supposed different, and also sufficiently far, from unity. Therefore, the system matrices can be decomposed to isolate the resonant sub-system (composed of the TUU dofs)

\[
G_0 = \omega^2 \begin{pmatrix} I & 0 \\ 0 & 1 \end{pmatrix}, \quad G_1 = \omega^2 \begin{pmatrix} 0 & g_1^T \\ g_1 & G_1^n \end{pmatrix}, \quad \text{where} \ G_1^n = \begin{pmatrix} d_\theta & c_\theta & c_\theta \\ c_\theta & d_{u1} & 0 \\ c_\theta & 0 & d_{u2} \end{pmatrix}, \quad g_1 = \begin{pmatrix} c_v \\ 0 \\ 0 \end{pmatrix}
\]

and the following \( \epsilon \)-order auxiliary parameters have been introduced

\[
d_\theta = 2 \beta \left( \frac{\rho_0}{\chi} \right)^2, \quad d_{u1} = -\sigma, \quad d_{u2} = \eta - \sigma, \quad c_\theta = \frac{\beta \rho_0}{2 \chi}, \quad c_v = \frac{\alpha \chi}{\chi \omega^2} \beta
\]

From an engineering viewpoint, \( d_{u1,2} \) and \( d_\theta \) can be recognized as (small) cable and deck disorder terms, respectively, while \( c_\theta \) can be identified as a (weak) cable-deck coupling term.

Employing a multi-parameter perturbation method suited to deal with multi-degrees-of-freedom nearly-resonant systems [6],[8], the asymptotic expansion (at the first \( \epsilon \)-power) of the eigenvalue matrix, uniformly valid in each parameter space direction, can be achieved

\[
\Lambda = \Lambda_0 + \epsilon \Lambda_1, \quad \Lambda_0 = \text{diag}(\omega^2, \omega^2, \omega^2, \Gamma^2 \omega^2), \quad \Lambda_1 = \text{diag}(\lambda_{11}^r, \lambda_{12}^r, \lambda_{13}^r, 0)
\]

that is found to well-approximate the exact eigensolution in the resonant region (Figure 3). The parametric form of the resonant eigenvalues \( \lambda_{11}^r = \omega^2 + \epsilon \lambda_{11}^r \) is reported in the Appendix.

Figure 3. Exact/asymptotic resonant eigenvalues versus (a) \( c_\theta \) for \( (c_v, d_\theta, d_{u1}, d_{u2}) = (4/25, 1/20, 0, 1/20) \), (b) \( d_\theta \) for \( (c_v, d_{u1}, c_\theta, d_{u2}) = (4/25, 0, 1/40, 1/20) \), (c) \( d_{u2} \) for \( (c_v, d_{u1}, d_\theta, c_\theta) = (4/25, 0, 1/20, 1/40) \).
If the eigenvector matrix is decomposed \( \Phi = [\Phi' | \Phi^t] \), where the 4-by-3 sub-matrix \( \Phi' \) collects the three resonant eigenvectors and the 4-by-1 sub-matrix \( \Phi^t \) collects the single non resonant mode, the first \( \varepsilon \)-power asymptotic expansion of the eigenvectors, shown in the Appendix, highlights the peculiar structure of the resonant sub-matrix

\[
\Phi^r = \Phi^0 + \varepsilon \Phi^1 = \begin{bmatrix} 0 \\ \Phi^0_r \end{bmatrix} + \varepsilon \begin{bmatrix} \Phi^1_m \\ 0 \end{bmatrix}
\]

(15)

in which the \( TUU \) modal components compose the 0th-order essential part \( \Phi^0_m \) (\( TUU \) dofs), while the \( V \) modal component composes the 1st-order complementary part \( \Phi^1_m \) (\( V \) dof).

### 4. Aeroelastic Stability Analysis

The viscous damping matrix \( D = Q^{-T} C Q^{-1} \) replicates the internal structure of the stiffness matrix, to synthetically account for the dissipation inherent to each structural connection

\[
D = \begin{pmatrix} \zeta_v & d^T \\ d & D^n \end{pmatrix}, \quad \text{where } D^n = \begin{pmatrix} \zeta_\theta & \zeta_c & \zeta_e \\ \zeta_c & \zeta_u & 0 \\ \zeta_e & 0 & \zeta_u \end{pmatrix}, \quad d = \begin{pmatrix} 0 \\ \zeta_b \end{pmatrix}
\]

(16)

where the terms \( \zeta_b \) and \( \zeta_e \) may simulate the passive control action of viscous dampers coupling the \( VUU \) and \( TUU \) dofs, often employed to mitigate the cable vibrations in suspended bridges.

Consistently with the ordering of the elasto-geometric parameters which govern the modal analysis, the structural damping and the aerodynamic terms can be ordered by setting

\[
\zeta_v = \varepsilon \zeta_v, \quad \zeta_\theta = \varepsilon \zeta_\theta, \quad \zeta_c = \varepsilon \zeta_c, \quad \zeta_u = \varepsilon \zeta_u, \quad \beta_A = \varepsilon \beta_A, \quad r_A = \varepsilon r_A, \quad \alpha_A = \varepsilon \alpha_A, \quad k_{\alpha} A = \varepsilon \tilde{k}_{\alpha} A
\]

(17)

according to which, the small to mid-range of wind flow velocities \( u_A \) can be investigated. All the aerodynamic coefficients are supposed to be \( O(1) \), according to experimental measurements.

Employing again the mass matrix decomposition \( M = Q^T \hat{Q} \), the equivalent aerodynamic stiffness \( G_A = Q^{-T} K_A Q^{-1} \), aerodynamic damping \( D_A = Q^{-T} C_A Q^{-1} \) and structural damping matrix, after plain substitutions and expansion to the first \( \varepsilon \)-power, read

\[
G_A = \varepsilon G_{A1} + O(\varepsilon^2), \quad D_A = \varepsilon D_{A1} + O(\varepsilon^2), \quad D = \varepsilon D_1 + O(\varepsilon^2)
\]

(18)

where the absence of zeroth \( \varepsilon \)-order terms can be remarked, confirming that the aerodynamic forces do not significantly alter, but can only perturb the system modal properties, namely the internal resonance conditions. Again, the first \( \varepsilon \)-order aerodynamic matrices can be properly decomposed to separate the terms acting on the \( TUU \) resonant degrees-of-freedom

\[
G_{A1} = \begin{pmatrix} 0 & g_{A1}^T \\ 0 & G_{A1}^n \end{pmatrix}, \quad D_{A1} = \begin{pmatrix} d_{A1}^m & d_{A1}^n \end{pmatrix}^T
\]

(19)

where the sub-matrices read

\[
d_{A1}^m = \zeta_v, \quad G_{A1}^n = \begin{pmatrix} g_{\theta} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad D_{A1}^n = \begin{pmatrix} \zeta_\theta & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad d_{A1}^m = \begin{pmatrix} \zeta_\theta v & 0 \\ 0 & 0 \end{pmatrix}, \quad d_{A1}^n = \begin{pmatrix} 0 \\ 0 \end{pmatrix}
\]

(20)

and the following \( \varepsilon \)-order auxiliary parameters have been introduced (tilde omitted)

\[
g_{\theta} = \frac{1}{2} k_{\alpha} A u_A \beta_A u_A c_m', \quad g_{\theta v} = \frac{1}{2} k_{\alpha} A u_A^2 c_m', \quad \zeta_v = \frac{1}{2} k_{\alpha} A (c_d + c'_d), \quad \zeta_{\theta v} = \frac{1}{2} k_{\alpha} A \beta_A c_m'
\]

(21)
4.1 Reduced-order model

Employing a classic Galerkin procedure, the displacement vector \( \mathbf{u} \) is expressed as a linear combination of the three resonant modes through the change-of-variable \( \mathbf{u} = \Phi^r \mathbf{q} \). Therefore, a reduced-order model, defined in the modal amplitudes \( \mathbf{q} \) and valid in the resonant regions of the parameter space, is obtained. The model response is governed by the equation of motion

\[
\ddot{\mathbf{M}} \mathbf{q} + \dot{\mathbf{D}} \mathbf{q} + \mathbf{G} \mathbf{q} = 0
\]

(22)

where the mass, stiffness and damping matrices follow from the coordinate change

\[
\ddot{\mathbf{M}} = (\Phi^r)^T \Phi^r, \quad \dot{\mathbf{D}} = \varepsilon (\Phi^r)^T (\mathbf{D} + \mathbf{D}_1^\varepsilon) \Phi^r, \quad \mathbf{G} = \ddot{\mathbf{G}}_0 + \varepsilon \mathbf{G}_1 = \omega^2 (\Phi^r)^T \Phi^r + \varepsilon (\Phi^r)^T (\omega^2 \mathbf{G}^n_1 + \mathbf{G}^n_1) \Phi^r
\]

(23)

According to the perturbation solution (15) and omitting the superscript \( r \) for the sake of simplicity, it can be demonstrated that the matrices can be consistently expressed as

\[
\ddot{\mathbf{M}} = (\Phi^n_0)^T \Phi^n_0, \quad \dot{\mathbf{D}} = \varepsilon (\Phi^n_0)^T (\mathbf{D}^n_0 + \mathbf{D}^n_{A1}) \Phi^n_0 \quad \text{and} \quad \mathbf{G} = \ddot{\mathbf{G}}_0 + \varepsilon \mathbf{G}_1 = \omega^2 (\Phi^n_0)^T \Phi^n_0 + \varepsilon (\Phi^n_0)^T (\omega^2 \mathbf{G}^n_1 + \mathbf{G}^n_1) \Phi^n_0
\]

(24)

This result evidences a key-issue, which is worth of the following mechanical interpretation. The perturbation solution of the classical modal problem (12)-(15) demonstrates how the free undamped dynamics of the three resonant \( TUU \) dofs is quasi-independent of the forth dof, as may be expected. Moreover, the 3\( dofs \) reduced-order model (22), defined in the amplitudes of the classic (real) resonant modes to describe the aerodynamics of the resonant \( TUU \) subsystem, possesses the same (complex) eigenvalues of the initial 4\( dofs \) model (6), if the forth non-resonant dof is simply neglected. From the mathematical viewpoint, this statement can be immediately proved by pre-multiplying and post-multiplying Eq.(22) by \((\Phi^n_0)^T\) and \((\Phi^n_0)^{-1}\), respectively. In practice, the first-order perturbation analysis filters out the coupling between the resonant \( TUU \) sub-system and the remaining non-resonant dof. A major remark is that, since the perturbation analysis excludes the \( RV \) resonance (by setting \( \Gamma \) far from unity), the non-symmetric couplings, i.e. the aeroelastic stiffness and damping terms \( g_{\theta\theta}, \xi_{\theta\theta}, \xi_{\thetav} \) are filtered out, ensuring that the resonant \( TUU \) sub-system is non-defective.

4.2 Critical conditions

According to the above considerations, the local aeroelastic stability analysis for the resonant \( TUU \) sub-system can be performed tracking the parameter-dependent loci of the eigenvalues \( \lambda_{1,2,3} \) (and their complex conjugates) which satisfy the characteristic equation

\[
\det[(\lambda^2 I + \lambda (\mathbf{D}^n_0 + \mathbf{D}^n_{A1}) + \omega^2 (\mathbf{G}^n_1 + \mathbf{G}^n_{A1})] = 0
\]

(26)

where all the \( \epsilon \)-order terms are between parentheses. The incipient instability arises for the critical condition \( Re(\lambda)=0 \), corresponding to zero real part of the \( i \)-th eigenvalue (namely \( Hopf \) bifurcation). The critical wind velocity is the lowest destabilizing \( u_A \)-value. Resorting to the numerical solution of the characteristic equation, preliminary parametric analyses have been performed in the three-dimensional parameter space spanned by the wind velocity \( u_A \) and the small \((\sigma,\varepsilon_c)\)-values, with a twofold aim: first, to determine the role played by the two resonant secondary systems, evaluating how this role can be also modified by a small internal detuning (due to the \( cable disorder \) \( \sigma \)) and, second, to check whether a dissipative coupling among the \( TUU \) (described by the \( \xi \)-parameter) may have a positive effect in moving the instability threshold to higher \( u_A \)-values. From a wider perspective, these issues may attract a certain engineering interest focused on the suitability to neglect resonant, though non-massive, hanger cables in aerodynamic studies of suspended bridges, and on the evaluation of the concurrent efficacy of viscous dampers, actually designed to mitigate the cable vibrations, in protecting also the bridge deck against wind-induced instabilities.
Figure 4. Aeroelastic stability analysis: (a) stability boundaries in the \((u_A, \sigma)\)-plane for \(\zeta_c=0\), (b), (c) eigenvalue loci versus the wind velocity \(u_A\) for different values of the cable disorder \(\sigma\) \((\omega=1, \beta=1/20, \alpha=2, \eta=0, \chi=1/8, \beta_A=1/20, k_A=1/40, r_A=1/40, c'_m=2.46)\).

Figure 4a illustrates the aerodynamic instability boundaries in the \((u_A, \sigma)\)-plane in absence of viscous coupling \((\zeta_c=0)\), and fixed values of all the remaining parameters. The geometric and mechanical parameter are referred to typical bridge structures, while the aerodynamic coefficients have been quantitatively assessed according to [9]. The curve of the critical wind velocity, corresponding to the stability boundary \(\mathcal{H}_1\), is almost independent of the cable disorder \(\sigma\). The eigenvalue loci corresponding to ordered cables \((\sigma=0)\) are shown in Figure 4b. The slight difference of the imaginary parts for \(u_A=0\) (related to the natural frequencies) depends on the small deck disorder \(d_\theta\). Increasing the wind velocity \(u_A\), the real (imaginary) part of a single eigenvalue grows up linearly (quadratically), and finally reaches the critical condition for \(u_A\approx 1.6\). The instability boundary \(\mathcal{H}_1\) is sensitive to the cable disorder only for a narrow negative value-range, around \(\sigma=-0.4\), where the curve exhibits a short and sharp right-oriented tongue, bending downwards. Therefore, since the loss of stability is moved towards higher wind velocities \(u_A\), and though the results may be not exhaustive, it can be pointed out that resonant or nearly-resonant cables do not induce dangerous effects, since their small mass can improve but not worsen the deck instability. With respect to the tongue-region, the local increment of the critical velocity can be related to the combination of the cable disorder \(\sigma\) and the \(u_A\)-dependent aerodynamic stiffness. Indeed, the imaginary part of two eigenvalues mainly depends on the cable disorder \(\sigma\), whose positive (negative) values have a softening (stiffening) effect. On the contrary, the imaginary part of the third eigenvalue systematically grows up, only due to the stiffening effect of increasing wind velocities \(u_A\). Thus, a balance between the two joined effects can occur only in the \(\sigma<0\) half-plane, if the increasing structural detuning caused by the cable disorder \(\sigma\) is compensated by the growing aerodynamic stiffness. Actually, a perfect balance (referred to as exact aerodynamic resonance) cannot occur, since the two approaching loci of the eigenvalue imaginary part undergo a strict veering phenomenon. Simultaneously, the two loci of the corresponding real part bend towards each other, undergoing instead a crossing phenomenon. Despite this peculiar behaviour is governed by the strong interaction between the imaginary parts of the eigenvalues, it causes the distortion of the instability boundary \(\mathcal{H}_1\) when it intersects the crossing locus of the real
parts of the eigenvalues $Re(\lambda_i) = Re(\lambda_j)$, marked by the black dotted curve $\mathcal{R}$. The distortion of the instability boundary $\mathcal{H}_1$ determines the local convexity of the stable region. Consequently, particular $\sigma$-values experiences three alternate change-in-sign of the eigenvalue real part, meaning that increasing wind velocities determine a fast destabilization-stabilization-destabilization process, as illustrated in Figure 4c. At higher wind velocities $u_A$ and negative $\sigma$-values, additional boundaries exist, close to the $\mathcal{R}$-curve. The instability boundary marked by the curve $\mathcal{H}_2$ corresponds to zero real part of a second eigenvalue which, for increasing $u_A$-values, becomes unstable. The instability boundary marked by the curve $\mathcal{H}_3$ corresponds again to zero real part of the first eigenvalue which, for increasing $u_A$-values, reverts to be stable. The codimension-1 instability boundaries $\mathcal{H}_1, \mathcal{H}_2, \mathcal{H}_3$, cross each other (together with the locus $\mathcal{R}$) in the codimension-2 point $\mathcal{T}$, associated to zero real part of two eigenvalues $Re(\lambda_i) = Re(\lambda_j) = 0$. However, the point $\mathcal{T}$ refers to a pair of eigenvalues moving in opposite directions for slightly-varying $u_A$-values, i.e., one incipiently unstable and the other incipiently stable. According to the discussion, the $(u_A, \sigma)$-plane can be decomposed in the stable region $SSS$ ($Re(\lambda_1) < 0$, $Re(\lambda_2) < 0$, $Re(\lambda_3) < 0$) and the two unstable regions $SUU$ ($Re(\lambda_1) > 0$, $Re(\lambda_2) > 0$, $Re(\lambda_3) > 0$), $SSU$ ($Re(\lambda_1) < 0$, $Re(\lambda_2) > 0$, $Re(\lambda_3) > 0$).

Figure 5a illustrates the aerodynamic instability boundaries in the $(u_A, \sigma)$-plane in presence of viscous coupling ($\zeta_c = 1/1000$). From the comparison with the previous case (the underlying gray curves), it can be remarked that the viscous coupling moves rightwards all the instability boundaries, enlarging the stable regions (in this case the stable area increases of about 60%) without qualitatively changing the structure of the stability chart. From an engineering viewpoint, this results suggest that additional viscous dampers acting on the cable-deck differential velocity may have a beneficial effect in the prevention of the aeroelastic instability through a significant increment of the critical wind velocity (see also the eigenvalues loci in Figure 5b). A minor difference can be recognized in the tongue-shaped distortion of the $\mathcal{H}_1$ instability boundary, which appears simultaneously enlarged and deformed. The deformation effect, in particular, ends up to extend the negative $\sigma$-value range in which increasing wind velocities $u_A$ may lead to the fast destabilization-stabilization-destabilization process already described (see also the eigenvalues loci in Figure 5c).

![Figure 5. Aeroelastic stability analysis: (a) stability boundaries in the $(u_A, \sigma)$-plane for $\zeta_c = 1/1000$, (b), (c) eigenvalue loci versus the wind velocity $u_A$ for different values of the cable disorder $\sigma$ ($\omega = 1$, $\beta = 1/20$, $\alpha = 2$, $\eta = 0$, $\chi = 1/8$, $\beta_A = 1/20$, $k_A = 1/40$, $r_A = 1/40$, $c' = 2.46$).](image-url)
5. CONCLUSIONS

The aeroelastic stability of a multi-body system has been studied through a four-degree-of-freedom model, which synthetically describes the linearized sectional dynamics of particular suspended bridges with doubly-symmetric cross-section. The model simultaneously accounts for both the internal structural interactions among the deck section (principal body) and a pair of pre-tensioned hanger cables (secondary bodies), and the aerodynamic coupling between the vertical flexion and torsion components of the deck motion, owing to a stationary wind flow acting along the horizontal symmetry axis. The classic modal problem has been solved by a multi-parameter perturbation method, suited to asymptotically well-approximate the solution in different resonance conditions. The perturbation solution has been employed to reduce the model dimension, as a first-order approximation consistently filters out the non-resonant deck flexional motion when focus is made on the triple resonance among a deck torsional mode (global mode) and a pair of cable modes (local modes). The local stability analysis in the resonance region points out a rich bifurcation scenario, featured by two stability boundaries in the parameter space. When the aerodynamic and the structural detuning are almost balanced (aerodynamic resonance), a marked frequency veering causes a strong reciprocal interaction between the two boundaries. From an engineering viewpoint, the analysis of the critical wind velocity reveals that resonant or nearly-resonant light cables may improve the deck stability. Furthermore, the introduction of additional viscous dampers, acting on the cable-deck differential velocity and typically designed to passively control the cable vibration (in service condition), may have a beneficial effect in the prevention of the aeroelastic instability (limit state condition), through a significant increment of the critical wind velocity.

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REFERENCES

APPENDIX

The non-null coefficients $M_{ij}$ and $K_{ij}$ of the 4-by-4 matrices $M$ and $K$ in equation (3) read

$$
\begin{align*}
M_{11} &= 1, & M_{22} &= \chi^2, & M_{33} &= M_{44} = \rho^2, & K_{11} &= \omega_v^2, & K_{33} &= 2\mu_1^2, & K_{44} &= 2\mu_2^2, \\
K_{22} &= \omega_\gamma^2 \alpha^2 \beta^2 + (\beta + 2\delta + \beta \delta)(\mu_1^2 + \mu_2^2)\beta / \delta, & K_{12} &= \alpha \beta \kappa, & K_{23} &= \beta \mu_1^2, & K_{24} &= \beta \mu_2^2
\end{align*}
$$

(A1)

The non-null coefficients $K_{Aij}$ and $C_{Aij}$ of the 4-by-4 matrices $K_A$ and $C_A$ in equation (6) read

$$
\begin{align*}
K_{A12} &= \frac{1}{2} k_A u_A^2 c'_l, & K_{A22} &= \frac{1}{2} k_A u_A^2 \beta_A c'_m, & C_{A11} &= \frac{1}{2} k_A u_A (c_d + c'_l), \\
C_{A12} &= -\frac{1}{2} k_A u_A r_A (c_d + c'_l), & C_{A21} &= \frac{1}{2} k_A u_A \beta_A c'_m, & C_{A22} &= -\frac{1}{2} k_A u_A r_A \beta_A c'_m
\end{align*}
$$

(A2)

The non null coefficients $J_{ij}$ and $I_{ij}$ of the matrices $G_0$ and $G_1$ in equation (9) read

$$
\begin{align*}
J_{11} &= \omega_v^2, & J_{22} &= \omega_v^2 \alpha^2 \beta^2 \chi^2, & J_{33} &= \frac{2\mu_2^2}{\rho_0^2}, & J_{44} &= \frac{2\mu_2^2}{\rho_0^2}, \\
I_{22} &= \frac{4\beta \mu_2^2}{\chi^2}, & I_{44} &= \frac{2\mu_2^2}{\rho_0^2}, & I_{12} &= \frac{\alpha \beta \kappa}{\chi}, & I_{23} &= \frac{\beta \mu_2^2}{\chi \rho_0}, & I_{24} &= \frac{\beta \mu_2^2}{\chi \rho_0}
\end{align*}
$$

(A3)

Setting the cable disorder $\sigma=0$ for the sake of simplicity, the perturbation form of the three resonant eigenvalues in the matrix $\Lambda$, defined in (14), is

$$
\begin{align*}
\lambda_1^2 &= \omega^2 + \frac{1}{3} \epsilon \omega^2 (d_{u2} + d_\theta + 2\Delta \cos(\varphi)), \\
\lambda_{2,3}^2 &= \omega^2 + \frac{1}{3} \epsilon \omega^2 (d_{u2} + d_\theta - 2\Delta \sin(\frac{1}{6} \pi \pm \varphi))
\end{align*}
$$

(A4)

and the corresponding eigenvectors read

$$
\varphi_{0i}^m = \left(c_\theta \omega \chi \lambda_{1i}^*, c_\theta \omega^4, (\chi_{1i}^*)^2 - \omega^2 d_\theta \lambda_{1i}^* - c_\theta^2 \omega^4 \right)^T, \\
\varphi_{1i}^m = \left(c_\theta c_\psi \omega^2 \chi_{1i}^* \right) \left(1 - I^2 \chi_{1i}^* \right)
$$

(A5)

where the following auxiliary parameters have been introduced

$$
\Delta = \left(6c_\theta^2 + d_{u2} - d_{u2}d_\theta + d_\theta^2 \right)^{1/2}, \quad \varphi = \frac{1}{3} \cos \left( \frac{d_{u2} - 2d_\theta}{2(6c_\theta^2 + d_{u2} - d_{u2}d_\theta + d_\theta^2)^{3/2}} \right)
$$

(A6)